



The Convexity of Quadratic Maps and the Controllability of Coupled Systems

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The Convexity of Quadratic Maps and the Controllability of Coupled Systems

ABSTRACT

A quadratic form on \mathbb{R}^n is a map of the form $x \mapsto x^T M x$, where M is a symmetric $n \times n$ matrix. A *quadratic map* from \mathbb{R}^n to \mathbb{R}^m is a map, all m of whose components are quadratic forms. One of the two central questions in this thesis is this: when is the image of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a convex subset of \mathbb{R}^m ?

This question has intrinsic interest; despite being only a degree removed from linear maps, quadratic maps are not well understood. However, the convexity properties of quadratic maps have practical consequences as well: underlying every semidefinite program is a quadratic map, and the convexity of the image of that map determines the nature of the solutions to the semidefinite program.

Quadratic maps that map into \mathbb{R}^2 and \mathbb{R}^3 have been studied before (in (Dines, 1940) and (Calabi, 1964) respectively). The Roundness Theorem, the first of the two principal results in this thesis, is a sufficient and (almost) necessary condition for a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to have a convex image when $m \geq 4$, $n \geq m$ and $n \neq m + 1$. Concomitant with the Roundness Theorem is an important lemma: when $n < m$, quadratic maps from \mathbb{R}^n to \mathbb{R}^m seldom have convex images.

The second result in this thesis is a controllability condition for bilinear systems defined on direct products of the form $\mathcal{G} \times \mathcal{G}$, where \mathcal{G} is a simple Lie group. The condition is this: a bilinear system defined on $\mathcal{G} \times \mathcal{G}$ is not controllable if and only if the Lie algebra generated by the system's vector fields is

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Jamin Lebbe Sheriff

the graph of some automorphism of \mathfrak{g} , the Lie algebra of \mathcal{G} .

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TO MOM AND POP SHERIFF (NAZLI AND SAKAF)

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1

Introduction

This thesis solves two disparate problems that are bound by their common origin in the control theory of Nuclear Magnetic Resonance (NMR) spectroscopy.

The first half of this thesis is about quadratically controlled control systems. These are systems of the form $\dot{x}(t) = Q(u(t))$, where $x(t) \in \mathbb{R}^m$ is the state of the system, $u(t) \in \mathbb{R}^n$ its control (say, a piecewise continuous function), and where $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *quadratic map*, a map each of whose components are homogeneous polynomials of degree 2 in the n variables x_1, \dots, x_n , the coordinates of \mathbb{R}^n (the components of a map mapping into \mathbb{R}^m are the compositions of the map with \mathbb{R}^m 's coordinate projections; these are properly defined below). Quadratically controlled systems are of interest because they model NMR experiments that involve coupled spins in the presence of relaxation.

One question will preoccupy the first half of this thesis: when can every point in the reachable set of a quadratically-controlled system be reached using a

constant control? The search for an answer to that question winds through the theory of convex cones, arriving eventually at a generalization of the Hausdorff-Toeplitz theorem.

The second half of this thesis is about invariant systems on special types of Lie groups, those that are the direct product $\mathcal{G} \times \mathcal{G}$ of a simple Lie group \mathcal{G} with itself. These systems represent a crude simplification of the *ensemble controllability problem*, which underlies the analysis of NMR systems whose magnetic fields possess spatial inhomogeneities.

The controllability of systems on Lie groups is a well-trodden subject. Controllability criteria for invariant systems on Lie groups exist, and are well-known [7]. However, by restricting its attention to a specific type of Lie group, this thesis arrives at a controllability condition that is commensurably more precise than existing conditions for general groups.

For clarity, the two problems alluded to above are now described explicitly. They are given in their barest forms, the ones they take when untethered from their roots in NMR spectroscopy.

Problem 1. Let $\pi_i : \mathbb{R}^m \rightarrow \mathbb{R}$ denote the i^{th} coordinate projection map $(x_1, \dots, x_m) \mapsto x_i$. Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function for which $\pi_i \circ Q : \mathbb{R}^n \rightarrow \mathbb{R}$ is a homogeneous quadratic polynomial for each $i \in \{1, \dots, m\}$. Find necessary and sufficient conditions on Q under which the image of Q is a convex subset of \mathbb{R}^m .

Problem 2. Let \mathfrak{g} be a simple Lie algebra, and let \mathfrak{k} be a subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$, the direct sum of \mathfrak{g} with itself. Find necessary and sufficient conditions on \mathfrak{k} under which \mathfrak{k} is a *proper* subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$.

This thesis solves both Problem 1 and Problem 2. Part I of this thesis delivers a solution to Problem 1. The solution goes by the name of The Roundness Theorem (Theorem 2.2.4); it is a necessary and sufficient condition for a large class of the type of function described in Problem 1 to be convex. A solution to Problem 2, the so-called Automorphism Condition (Theorem 8.1.1) is given in of Part II of the thesis.

The remainder of this introduction is devoted to the exposition of Problems 1 and 2, with a focus on their evolution from problems in NMR spectroscopy. The introduction begins with a cursory review of the theory of magnetic resonance. From there it introduces quadratically controlled systems, beginning with one that describes the dynamics of coupled nuclear spins. Section 1.2 establishes a relationship between the reachable sets of such systems and the convexity properties of the quadratic maps that define them. This is a basic but important result; it provides the impetus for a study of quadratic maps themselves.

The chapter then moves on to the controllability of invariant systems on direct products of simple groups. For brevity, and for reasons to be explained later (see Section 1.3), these systems will henceforth be referred to as *control-coupled* systems. Section 1.3 introduces control-coupled systems as precursors to the *ensemble controllability problem*, which is discussed briefly in Section 1.3 and at length in [29]. The discussion in Section 1.3 begins simply, starting with systems on $SO(3) \times SO(3)$ that represent that evolution of two coupled spins, but eventually culminates by posing Problem 2 in general terms.

1.1 A BRIEF INTRODUCTION TO NMR SPECTROSCOPY

This section presents a brief introduction to nuclear magnetic resonance spectroscopy. A comprehensive account can be found in [30].

Remark 1.1.1. This section makes free use of the notion of spin without making any attempt to define or explain it. An introduction to quantum mechanics and the nature of spin can be found in [12]. ■

NMR spectroscopy is a well-established technique for determining the nature of matter. It can be used to discern the chemical, structural and electronic properties of molecules ranging from simple amino acids consisting of a handful of atoms, to complex proteins comprised of many thousands of atoms. Applications of NMR spectroscopy can be found in a variety of settings, from medical imaging [33] to the petroleum industry [37].

In the most basic NMR experiment, an ensemble of non-interacting nuclear spins (sometimes referred to as the *sample*) is exposed to two magnetic fields: one a very strong, spatially and temporally constant field (denoted by B_o), and the other a weaker, time-varying one (denoted by $B_{rf}(t)$). The latter is typically a radio-frequency (RF) field that is orthogonal to B_o ; it plays a decisive role in determining the outcome of the NMR experiment.

The nucleus of an atom has spin, and that spin endows the nucleus with a magnetic moment. An ensemble of atoms thus has a bulk magnetic moment; this is the sum of the magnetic moments of the individual nuclei in the ensemble. In non-magnetic materials, this sum is ordinarily zero. However, when subject to the foregoing combination of magnetic fields, most substances (those whose constituent atoms have non-zero nuclear spin) will develop a non-zero bulk magnetic moment. The moment thus induced will create a current in a judiciously placed receiver. The resulting signal is referred to as the *free induction decay* (FID), or more plainly as the *NMR signal*. If sampled, recorded, and properly analyzed, an NMR signal can impart characteristics of the material being studied.

The bulk magnetic moment thus induced in the sample is denoted by M and obeys the *Bloch equation*,

$$\dot{M}(t) = M(t) \times B(t) - R \cdot (M(t) - M_o) \quad (1.1.1)$$

Here, $M(t) \in \mathbb{R}^3$ is the bulk magnetic moment of the sample, and $B(t) = B_o + B_{rf}(t) \in \mathbb{R}^3$ the magnetic field to which the substance is subjected (the sum of the constant field B_o and the RF-field $B_{rf}(t)$). R is a diagonal matrix with positive entries that represents processes by which M decays to a constant equilibrium value $M_o \in \mathbb{R}^3$; these processes are collectively referred to as relaxation. The Bloch equation says, roughly, that the magnetic moment $M(t)$ precesses around the magnetic field $B(t)$ while at the same time converging towards M_o .

Remark 1.1.2. The coordinates of \mathbb{R}^3 are usually chosen in such a way that B_o is

parallel to the z -direction: $B_o = (0, 0, \|B_o\|)$. $B_{rf}(t)$, being orthogonal to B_o , can then be written as $B_{rf}(t) = (u(t), v(t), 0)$, where u and v are parameters describing the time dependence of B_{rf} . ■

The efficacy of an NMR experiment depends on a number of factors, but only one of them can be controlled by the experimenter, namely the time-dependence of the RF-field B_{rf} . This raises the question of how to design the parameters u and v in a way that best achieves the objectives of the experiment, which could include minimizing the duration of the experiment or minimizing the power consumed over that duration. The problem of designing the RF fields for NMR spectroscopy is usually referred to as *pulse sequence design*.

For the purposes of this thesis, the origins of the Bloch equation are less important than the fact that the Bloch equation is a differential equation with parameters. This fact makes the problem of pulse sequence design a natural subject for the tools of control theory. The advent of control-theoretic techniques in NMR spectroscopy is a recent development, but has nevertheless yielded important theoretical insights and significant practical advances [23, 24]. The preponderance of those insights have been concerned with the optimal control of NMR systems; for example, time-optimality [23], or the design of pulse sequences that maximize certain coherence transfer efficiencies [24]. The focus of this thesis is on a different aspect of those control systems that arise in NMR spectroscopy: their controllability.

1.2 QUADRATICALLY CONTROLLED SYSTEMS

The descriptive power of the Bloch equation is limited by the fact that the Bloch equation applies only to ensembles of noninteracting nuclear spins. In real experiments, the nuclear spins in a collection of atoms will interact with one another; in a word, they are coupled.

The Bloch equation does not describe the behaviour of coupled spins. Nevertheless, the dynamics of coupled spin systems are still governed by a linear, ordinary differential equation in which the components of the rf-field B_{rf} appear

as parameters (this is the Liouville-von Neumann equation [11]). The Liouville-von Neumann equation will not be discussed here. For, like that of the Bloch equation, the amenability of the Liouville-von Neumann equation to the methods of control theory is more important than the equation itself.

The authors of the papers [24] and [40] studied the dynamics of two coupled spins undergoing an NMR experiment. In those papers, after some algebraic manipulations, the Liouville-von Neumann equation was reduced to this,

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \xi u_1(t)^2 - u_1(t)u_2(t) \\ u_1(t)u_2(t) - \xi u_2(t)^2 \end{pmatrix} \quad (1.2.1)$$

Here, $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ represents the expected values of certain spin-angular momentum operators. The functions u_i are controls that are implicitly related to the components of the rf-field B_{rf} . ξ is a constant representing the relative strengths of certain microscopic relaxation processes [40].

The principal concern of the paper [24] was this:

Question 1.2.1. *If the initial state of the system (1.2.1) is $(1, 0)$, how close (with respect to the standard norm on \mathbb{R}^2) can the state be brought to the point $(0, 1)$ using bounded controls that have at most a finite number of discontinuities?*

This question embodies the problem of maximizing coherence transfer between spins, an important aspect of many NMR experiments [24].

Within the focus of the paper [40] was a similar question:

Question 1.2.2. *Of the states that can be reached along system (1.2.1) from a given initial state using the set of controls described in Question 1.2.1, how many can be reached using controls u_1, u_2 that are constant?*

This question derives its importance from the simplicity of constant controls; they are, simply, easier to implement than are controls that vary in time.

Both questions, Question 1.2.1 and Question 1.2.2, involve the nature of the reachable sets of system (1.2.1). These can be understood with the help of a broader perspective.

System (1.2.1) is an example of a *quadratically controlled system*, a system of the form

$$\dot{x}(t) = Q(u(t)) \quad (1.2.2)$$

where $x(t) \in \mathbb{R}^m$, $u(t) \in \mathbb{R}^n$, and $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *quadratic map*,

Definition 1.2.3. A function $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **quadratic map** if it has the form

$$x \mapsto \begin{pmatrix} \langle x, M_1 x \rangle \\ \langle x, M_2 x \rangle \\ \vdots \\ \langle x, M_m x \rangle \end{pmatrix}$$

for every $x \in \mathbb{R}^n$, where \langle, \rangle denotes the standard inner product on \mathbb{R}^n and the M_i are $n \times n$ symmetric matrices. Q is said to be defined by the M_i .

Example 1.2.4. The quadratic map in the quadratically controlled system (1.2.2) is the one defined by the 2×2 matrices

$$\begin{pmatrix} \xi & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & \xi \end{pmatrix}$$

Remark 1.2.5. In the preamble to this chapter, quadratic maps were defined as maps from \mathbb{R}^n to \mathbb{R}^m , each of whose components is a homogeneous, degree 2 polynomial on \mathbb{R}^n . This definition is equivalent to Definition 1.2.3. However, Definition 1.2.3 is more convenient. Many properties of a quadratic map Q can be neatly expressed in terms of the subspace of Sym_n (the space of $n \times n$ symmetric matrices) spanned by Q 's defining matrices $M_i \in \text{Sym}_n$. Nevertheless, quadratic maps will still occasionally be written as vectors of quadratic polynomials, their defining matrices M_i left implicit. ■

From the more general point of view suggested by Definition 1.2.3, Questions 1.2.1 and 1.2.2 become aspects of a broader inquiry: what does the reachable set

of a quadratically controlled system look like? As a first step towards describing the reachable set of a quadratically controlled system, note that system (1.2.2) can be written as

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} Q(u(\tau)) \, d\tau \quad (1.2.3)$$

where $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$. If the control $u : [t_0, t_1] \rightarrow \mathbb{R}^n$ is constant and equal to $u_0 \in \mathbb{R}^n$, then,

$$\begin{aligned} x(t_1) &= x(t_0) + (t_1 - t_0)Q(u_0) \\ &= x(t_0) + Q(\overline{u_0}) \end{aligned}$$

where $\overline{u_0} = \sqrt{t_1 - t_0} \cdot u_0$. The final equality follows from the homogeneity of Q : $Q(av) = a^2Q(v)$ for all $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$. It follows that the set of points reachable from $x(t_0)$ using only constant controls in system (1.2.2) is the sum of $x(t_0)$ with the image of Q (that is, the set of points $\{x(t_0) + Q(u) : u \in \mathbb{R}^n\}$).

Suppose now that $u : [t_0, t_1] \rightarrow \mathbb{R}^n$ is a step function, a function with the form $u(t) = \sum_{i=1}^p v_i \chi_{E_i}(t)$ where p is some positive integer, $v_i \in \mathbb{R}^n$, $E_i \subset [t_0, t_1]$ and χ_{E_i} the indicator function of E_i . It is assumed that for $i = 1, \dots, p-1$, E_i is a left-closed, right-open interval, and that E_p is simply a closed interval (containing t_1). It is also assumed that the subsets E_i are ascending (if $i < j$, then $x < y$ for all $x \in E_i, y \in E_j$), and that they form a partition of $[t_0, t_1]$ (they are disjoint and their union is $[t_0, t_1]$).

For the preceding choice of control, the k^{th} component of the right side of (1.2.3) is

$$x(t_0)_k + \int_{t_0}^{t_1} \langle u(\tau), M_k u(\tau) \rangle \, d\tau$$

where $M_k \in \text{Sym}_n$ is the k^{th} matrix defining Q . This is equal to

$$\begin{aligned} x(t_o)_k + \sum_{i,j=1}^p \langle v_i, M_k v_j \rangle \cdot \int_{t_o}^{t_1} \chi_{E_i}(\tau) \chi_{E_j}(\tau) \, d\tau \\ = x(t_o)_k + \sum_{i=1}^p \langle v_i, M_k v_i \rangle \cdot \lambda(E_i) \end{aligned} \quad (1.2.4)$$

where $\lambda(E_i)$ denotes the length (the Lebesgue measure) of the interval E_i .

Written as an equation in \mathbb{R}^m , equation (1.2.4) becomes,

$$x(t_1) = x(t_o) + \sum_{i=1}^p \lambda(E_i) \cdot Q(v_i) \quad (1.2.5)$$

The indexed sum in (1.2.5) lies in the convexification of the image, $\text{Im } Q$, of Q (a review of the theory of convexity will be provided in Section 3.3 of the next chapter). Indeed, denoting $\lambda(E_i)$ by λ_i and $\sum_{j=1}^p \lambda_j$ by Λ ,

$$\begin{aligned} \sum_{i=1}^p \lambda_i \cdot Q(v_i) \\ = \Lambda \sum_{i=1}^p \frac{1}{\Lambda} \lambda_i \cdot Q(v_i) \\ = \sum_{i=1}^p \frac{1}{\Lambda} \lambda_i \cdot Q(\bar{v}_i) \end{aligned} \quad (1.2.6)$$

where $\bar{v}_i = \sqrt{\Lambda} v_i$. The final equality follows again from the homogeneity of Q .

The expression (1.2.6) is a convex combination of the points $Q(v_i) \in \mathbb{R}^n$, $i = 1, \dots, p$, and therefore lies in the convexification of $\text{Im } Q$. If the set of admissible controls in system (1.2.2) is momentarily taken to be the set of step functions from $[t_o, t_1]$ to \mathbb{R}^n , it follows from (1.2.6) that the set of points reachable from $x(t_o)$ is the sum of $x(t_o)$ and the convexification of $\text{Im } Q$. If the set of admissible controls is enlarged to include those functions from $[t_o, t_1]$ to \mathbb{R}^n

that are Riemann integrable, then the set of points reachable from $x(t_0)$ is the sum of $x(t_0)$ and the *closure* of the convexification of $\text{Im } Q$.

Let Q_ξ denote the quadratic map underlying the quadratically controlled system (1.2.1). Question 1.2.1 can now be posed as this: *what is the minimum distance between $(0, 1)$ and the set $(0, 1) + \text{Cl}_{\mathbb{R}^2}(\text{Conv}(\text{Im } Q_\xi))$, where $\text{Cl}_{\mathbb{R}^2}$ denotes the closure in \mathbb{R}^2 and Conv the convexification. Similarly, Question 1.2.2 becomes this: *when is the image of Q_ξ equal to its convexification?**

This leads at last to the first of the two principal questions in this thesis. It was stated above as Problem 1, but is restated here using the updated terminology of this section.

Problem 1. When is the image of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a convex subset of \mathbb{R}^m ?

This question, its history, and this thesis' solution to it, are explored in Chapter 2.

1.3 CONTROL-COUPLED SYSTEMS

The second problem in this thesis is about systems on direct products of Lie groups in which the dynamics in any given factor are independent of the state of the other factor. The dynamics in each factor of the direct product are coupled to those in the other factor only, if at all, by the controls; briefly, they are control-coupled. Control-coupled systems arise in NMR in a roundabout way, when studying the effects of amplitude dispersion, which are discussed below.

In an actual NMR experiment, the amplitude of the rf-field B_{rf} may, at any given time, vary from one point in the sample to another. This variation is referred to as *rf-inhomogeneity* or *amplitude dispersion*. Amplitude dispersion can have unwanted effects that the NMR experimentalist must compensate for. This section discusses those effects together with one particular approach to overcoming them.

Amplitude dispersion can be expressed mathematically by writing the rf-field

as $B_{rf} : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$, $B_{rf}(r, t) = A(r) \cdot B_{rf}^\circ(t)$ where $A : \mathbb{R}^3 \rightarrow [0, \infty)$ is a continuous function representing the position dependence of the amplitude of B_{rf} , and where $B_{rf}^\circ : \mathbb{R} \rightarrow \mathbb{R}^3$, $B_{rf}^\circ(t) = (u(t), v(t), 0)$ is the intended, spatially uniform rf-field. Let $M_\varepsilon(t) \in \mathbb{R}^3$ denote the sum of the magnetic moments of all those spins in the sample that are situated at points $r \in \mathbb{R}^3$ at which $A(r) = \varepsilon$. If the sample consists of noninteracting spins, then M_ε evolves according to the Bloch equation with rf-field $\varepsilon B_{rf}^\circ(t)$,

$$\dot{M}_\varepsilon(t) = M_\varepsilon(t) \times [B_0 + \varepsilon B_{rf}^\circ(t)] - R \cdot (M_\varepsilon(t) - M_0) \quad (1.3.1)$$

Thus, rf-inhomogeneity causes the magnetic moments of spins in different parts of the sample to evolve according to different dynamics.

The bulk magnetic moment of the sample is the integral,

$$M(t) = \int_0^\infty M_\varepsilon(t) \, d\varepsilon \quad (1.3.2)$$

The NMR experimentalist must use the rf-field to manipulate the bulk moment M in a way that yields a fruitful NMR signal. This often involves having M point in a certain direction at a certain time. It follows from (1.3.2) that, when the rf-field suffers from amplitude dispersion, M can be made to point in a given direction at a given time by arranging for all of the M_ε to point in that same direction at that same time.

The problem with rf-inhomogeneity is now clear: the NMR experimentalist must maneuver *every* M_ε (of which there are uncountably many) using only the two controls u and v . The problem of controlling uncountably many control systems using only finitely many controls has come to be known as the *ensemble controllability problem* (ECP). Comprehensive accounts of the ECP can be found in [29] and [31]. In those papers, the ECP is treated as a control problem whose state space is a function space. As such, the ECP falls outside the purview of conventional control theory, which trades in problems whose state spaces are finite-dimensional. The focus of this section is on a problem that resembles the

ECP but is amenable to classical tools of control theory.

The following discourse will benefit from a more precise statement of the ECP than the one given above. To that end, suppose that $\|M_\varepsilon(o)\| = 1$ for all $\varepsilon \in [o, \infty)$, where $\|\cdot\|$ denotes the standard norm on \mathbb{R}^3 (this will not affect the generality of the forthcoming observations). In the absence of relaxation (that is, when $R = o$ in (1.3.1)), the norm $\|M_\varepsilon(t)\|$ is independent of t ; each M_ε evolves on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Under these assumptions, the ECP takes the following form,

The Ensemble Controllability Problem. *Let $F_i : (o, \infty) \rightarrow \mathbb{S}^2, i = o, 1$ be two functions. Are there functions $u, v : [o, \infty) \rightarrow \mathbb{R}$ for which the solutions to (1.3.1) satisfy $M_\varepsilon(o) = F_o(\varepsilon)$ and $M_\varepsilon(1) = F_1(\varepsilon)$?*

Less formally, the ECP asks whether each of the systems (1.3.1), indexed by $\varepsilon \in [o, \infty)$, can be simultaneously driven from a given collection of initial points $M_\varepsilon(o) = F_o(\varepsilon)$ to a given collection of endpoints $M_\varepsilon(1) = F_1(\varepsilon)$, using only the two controls u and v .

Of course, there is no hope of driving the ensemble of systems (1.3.1) (indexed by $\varepsilon \in (o, \infty)$) from one collection of points in \mathbb{S}^2 to another if the same cannot be done for an individual system ((1.3.1), for a fixed ε). Things are similarly desperate if any finite subcollection of the systems (1.3.1) cannot be jointly controlled. Thus, a natural antecedent to the ECP is the problem of determining the controllability of the following system,

$$\begin{pmatrix} \dot{M}_{\varepsilon_1}(t) \\ \vdots \\ \dot{M}_{\varepsilon_n}(t) \end{pmatrix} = \begin{pmatrix} M_{\varepsilon_1}(t) \times [B_o + \varepsilon_1 B_{rf}^o(t)] \\ \vdots \\ M_{\varepsilon_n}(t) \times [B_o + \varepsilon_n B_{rf}^o(t)] \end{pmatrix} \quad (1.3.3)$$

on the n -fold product $\mathbb{S}^2 \times \cdots \times \mathbb{S}^2$. Here, n is some natural number, and $\{\varepsilon_1, \dots, \varepsilon_n\} \subset (o, \infty)$ a collection of positive numbers. An expeditious exploration of (1.3.3) will lead to this thesis' second foundational question; the ideas essential to this pursuit can all be found in the cases $n = 1$ and $n = 2$ in

(1.3.3). There is some mention of Lie groups and Lie algebras in what follows. The necessary background material is summarized in Chapter 8 and can be found in [25].

When $n = 1$ in (1.3.3), the solution to (1.3.3) can be written as $M_{\varepsilon_1}(t) = \theta_{\varepsilon_1}(t)M_{\varepsilon_1}(o)$, where $\theta_{\varepsilon_1} : [o, \infty) \rightarrow SO(3)$ is a curve taking values in the group of orthogonal 3×3 matrices and satisfying

$$\begin{aligned}\dot{\theta}_{\varepsilon_1}(t) &= [\Omega_{B_o} + \varepsilon_1 u(t)\Omega_x + \varepsilon_1 v(t)\Omega_y] \theta_{\varepsilon_1}(t) \\ \theta_{\varepsilon_1}(o) &= \mathbb{I}_3\end{aligned}\tag{1.3.4}$$

Here, \mathbb{I}_3 is the identity map on \mathbb{R}^3 , and Ω_{B_o} , Ω_x and Ω_y are the following elements of $\mathfrak{so}(3)$, the space of skew-symmetric, 3×3 matrices,

$$\begin{aligned}\Omega_x &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ \Omega_y &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \Omega_{B_o} &= \begin{pmatrix} 0 & B_o & 0 \\ -B_o & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

Let $M_{\varepsilon_1}(o) \in \mathbb{S}^2$ be a fixed initial point. Given any $M_1 \in \mathbb{S}^2$, there exists a $\theta \in SO(3)$ satisfying $\theta M_{\varepsilon_1}(o) = M_1$. It follows that, when $n = 1$, (1.3.3) is controllable if (1.3.4) is controllable.

Equation (1.3.4) is an example of a bilinear, right-invariant control system on a compact matrix group, the general form of which is,

$$\dot{\theta}(t) = \left[X_o + \sum_{i=1}^m u_i(t)X_i \right] \theta(t)\tag{1.3.5}$$

Here, $\theta : [0, \infty) \rightarrow \mathcal{G}$ is a curve taking values in a compact matrix group \mathcal{G} , a compact subgroup of the group of linear isomorphisms of \mathbb{R}^n , for some n . The X_i ($i = 0, \dots, m$) are elements of \mathfrak{g} , the Lie algebra of \mathcal{G} , and the functions $u_i : [0, \infty) \rightarrow \mathbb{R}$ are controls. Controllability criteria for such systems were first established in [7] and [22]. According to those criteria, (1.3.5) is controllable if and only if the Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ generated by $\{X_0, \dots, X_m\}$ is equal to \mathfrak{g} itself.

The Lie subalgebra $\mathfrak{k} \subset \mathfrak{so}(3)$ generated by $\{\Omega_{B_0}, \Omega_x, \Omega_y\}$ is indeed $\mathfrak{so}(3)$ itself. It follows that (1.3.4) is controllable and therefore that (1.3.3) is controllable when $n = 1$. This, moreover, is true for any $\varepsilon_1 \in (0, \infty)$ (in fact, for any nonzero ε_1).

The foregoing approach to the controllability of (1.3.3) when $n = 1$ applies equally well to any value of n . In particular, when $n = 2$ in (1.3.3), the solution to (1.3.3) can be written as

$$\begin{pmatrix} M_{\varepsilon_1}(t) \\ M_{\varepsilon_2}(t) \end{pmatrix} = \begin{pmatrix} \theta_{\varepsilon_1}(t) & 0 \\ 0 & \theta_{\varepsilon_2}(t) \end{pmatrix} \begin{pmatrix} M_{\varepsilon_1}(0) \\ M_{\varepsilon_2}(0) \end{pmatrix} \quad (1.3.6)$$

where

$$\begin{pmatrix} \theta_{\varepsilon_1} & 0 \\ 0 & \theta_{\varepsilon_2} \end{pmatrix} : [0, \infty) \rightarrow SO(3) \times SO(3)$$

is a curve taking values in the group $SO(3) \times SO(3)$ (thought of here as the block diagonal subgroup of $SO(6)$ having two 3×3 blocks), and satisfying

$$\begin{aligned} & \begin{pmatrix} \dot{\theta}_{\varepsilon_1}(t) & 0 \\ 0 & \dot{\theta}_{\varepsilon_2}(t) \end{pmatrix} \\ &= [\Omega_{B_0 \oplus B_0} + u(t)\Omega_{\varepsilon_1 x \oplus \varepsilon_2 x} + v(t)\Omega_{\varepsilon_1 y \oplus \varepsilon_2 y}] \begin{pmatrix} \theta_{\varepsilon_1}(t) & 0 \\ 0 & \theta_{\varepsilon_2}(t) \end{pmatrix} \end{aligned} \quad (1.3.7)$$

where $\Omega_{B_0 \oplus B_0}$, $\Omega_{\varepsilon_1 x \oplus \varepsilon_2 x}$ and $\Omega_{\varepsilon_1 y \oplus \varepsilon_2 y}$ are the following elements of the Lie algebra

$$\mathfrak{so}(3) \oplus \mathfrak{so}(3),$$

$$\begin{aligned}\Omega_{B_o \oplus B_o} &= \begin{pmatrix} \Omega_{B_o} & o \\ o & \Omega_{B_o} \end{pmatrix} \\ \Omega_{\varepsilon_1 x \oplus \varepsilon_2 x} &= \begin{pmatrix} \varepsilon_1 \Omega_x & o \\ o & \varepsilon_2 \Omega_x \end{pmatrix} \\ \Omega_{\varepsilon_1 y \oplus \varepsilon_2 y} &= \begin{pmatrix} \varepsilon_1 \Omega_y & o \\ o & \varepsilon_2 \Omega_y \end{pmatrix}\end{aligned}$$

$(\mathfrak{so}(3) \oplus \mathfrak{so}(3))$ is thought of here as the set of block diagonal 6×6 matrices having two 3×3 skew-symmetric blocks).

Once again, (1.3.6) is controllable so long as (1.3.7) is controllable. And, again by the criteria of [7] and [22], (1.3.7) controllable if and only if the Lie subalgebra $\mathfrak{k} \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ generated by $\{\Omega_{B_o \oplus B_o}, \Omega_{\varepsilon_1 x \oplus \varepsilon_2 x}, \Omega_{\varepsilon_1 y \oplus \varepsilon_2 y}\}$ is equal to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ itself.

At this juncture, an explicit calculation of the subalgebra \mathfrak{k} and a verification of the equality $\mathfrak{k} = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ would suffice to show that (1.3.6) is controllable. However, neither of these steps is necessary. System (1.3.7) has features that distinguish it from generic bilinear, invariant systems, and which simplify the analysis of its controllability (those features will be identified later on, in Chapter 8).

The controllability of (1.3.7) is governed by a rule that does not involve the explicit calculation of the subalgebra \mathfrak{k} . The first step in finding that rule is to understand the structure of system (1.3.7).

System (1.3.6) is comprised of two bilinear, right-invariant systems on $SO(3)$, one corresponding to the upper 3×3 block in (1.3.7) and the other corresponding to the lower 3×3 block. Explicitly, the two systems are,

$$\dot{\theta}_{\varepsilon_1}(t) = [\Omega_{B_o} + \varepsilon_1 u(t) \Omega_x + \varepsilon_1 v(t) \Omega_y] \theta_{\varepsilon_1}(t) \quad (1.3.8)$$

$$\dot{\theta}_{\varepsilon_2}(t) = [\Omega_{B_o} + \varepsilon_2 u(t) \Omega_x + \varepsilon_2 v(t) \Omega_y] \theta_{\varepsilon_2}(t) \quad (1.3.9)$$

Each of the component systems (1.3.8) and (1.3.9) is separately controllable: the upper 3×3 block (1.3.8) can be driven between any two points in $SO(3)$, and the same is true of the lower block (1.3.9). This follows from the controllability of (1.3.4). However, in the full system (1.3.7), these component systems are controlled *simultaneously* by the controls u and v ; the systems are control-coupled.

It is impossible to deduce only from the individual controllability of the upper and lower subsystems (1.3.8) and (1.3.9) that the full system (1.3.7) is controllable. For example, if $\varepsilon_1 = \varepsilon_2$ in (1.3.7), every trajectory of the system will be bound to a coset of the diagonal subgroup of $SO(3) \times SO(3)$. The system will not be controllable in this case, even though both of its component systems are separately controllable. The central question that emerges in the analysis of the controllability of (1.3.7), then, is this,

Question 2. Given a bilinear, right-invariant system on $SO(3) \times SO(3)$ each of whose component systems are controllable, when is the full system controllable?

Question 2 rewards abstraction. Chapter 8 will explore the analogous question about systems on $\mathcal{G} \times \mathcal{G}$, where \mathcal{G} is a Lie group that shares one defining characteristic with $SO(3)$, namely its simplicity. The main result there, the Automorphism Condition (Theorem 8.1.1), is a necessary and sufficient condition for such systems to be controllable.

1.4 THE LAYOUT OF THIS THESIS

This thesis is organized as follows.

Chapter 2. Chapter 2 explores the history of inquiries into the nature of quadratic maps. It also states the first of this thesis' two main results, the Roundness Theorem.

Chapter 3. Chapter 3 establishes all of the mathematical background material that is needed for the proof of the Roundness Theorem. This includes basic facts about convex sets and cones, especially dual cones, as well as a smattering of

differential geometry.

Chapter 4. Chapter 4 paints a detailed picture of quadratic maps. It begins by laying out a number of properties of the image of a quadratic map; these will be essential for studying the convexification of the image. Chapter 4 includes a detailed treatment of two important classes of quadratic maps: definite maps and round maps. The image of a definite quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the union of the origin $o \in \mathbb{R}^m$ and a cone that lies in an *open* half-space of \mathbb{R}^m . This fact simplifies the analysis of the convexity of the image of Q . A round quadratic map is a type of definite map. Round quadratic maps are so named because of an important property: the boundary of the image of a round quadratic map is a smooth manifold.

Chapter 5. Chapter 5 proves the Roundness Theorem. The Roundness Theorem states first that roundness is a sufficient condition for the convexity of the image of a quadratic map. It also states that roundness is *almost* necessary for convexity: a definite quadratic map that is not round can be made nonconvex via arbitrarily small perturbations to the map. Chapter 5 also proves an important lemma, the Instrumental Lemma, which plays an instrumental role in proving one of the implications in the Roundness Theorem.

Chapter 6. Chapter 6 discusses the application of the Roundness Theorem to the quadratically-controlled system that was introduced in Section 1.2. This was the original impetus for the study of quadratic maps that led to the Roundness Theorem.

Chapter 7. Chapter 7 covers a few aspects of quadratic maps that are not covered in Chapters 4 and 5. These include indefinite and semidefinite quadratic maps, a brief discussion of a peculiar quadratic map and an elementary description of the convexification of the image of a quadratic map that is not used elsewhere in the thesis.

Chapter 8. Finally, Chapter 8 solves Problem 2. The chapter consists mostly of a proof of the Automorphism Condition and some generalizations thereof.

Part I

The Convexity of Quadratic Maps

2

Quadratic Maps at Large

This chapter is a survey of previous attempts to understand the convexity of quadratic maps. In the course of studying the history of quadratic maps, this chapter will introduce all of the concepts that are needed to properly state the Roundness Theorem. The chapter begins with a few illustrative examples.

Example 2.0.1. *The simplest example of a quadratic map from \mathbb{R}^n to itself is the one that squares the coordinates of its argument,*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} (x_1)^2 \\ \vdots \\ (x_n)^2 \end{pmatrix}$$

*This map will be referred to as the **standard quadratic map**. In the terms of Definition 1.2.3, the standard quadratic map is defined by the matrices $M_i = E_{ii}$,*

where E_{ii} is the $n \times n$ matrix with a 1 in the i^{th} diagonal and zeros elsewhere.

The image of the standard quadratic map is easy to describe: it is the positive orthant of \mathbb{R}^n . It is similarly easy to compute the preimage of a point in the image of the standard quadratic map: if $p \in \mathbb{R}^n$ is such a point, then the preimage of p under the standard quadratic map consists of the 2^n points in \mathbb{R}^n whose i^{th} coordinates are equal in absolute value to the positive square root of the i^{th} coordinate of p .

The standard quadratic map is deceptively simple; it does not capture the full range of behaviours that quadratic maps can exhibit. Over the course of this thesis, it will become clear that quadratic maps can be inscrutable.

Quadratic maps share a kinship with linear maps. Like those of linear maps, the components of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (the compositions $\pi_i \circ Q$, where π_i is the i^{th} coordinate projection of \mathbb{R}^m) are all homogeneous polynomials, all sharing the same degree (2, in the case of quadratic maps). Yet, despite being only a degree removed from linear maps, quadratic maps are poorly understood.

Whereas a linear map can be put into one of a number of normal forms from which properties of the map, its surjectivity for example, can be readily determined, there are few known ways to deduce any property of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ from the matrices $M_i \in \text{Sym}_n$ that define it.

The problem of extracting information about Q from the matrices M_i has received intermittent attention, the fruits of which are described in the next section. Every previous inquiry into the nature of quadratic maps has focused on the same question, namely Problem 1 of Chapter 1, which is stated here in more general terms,

Problem 1. When is the image of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a convex subset of \mathbb{R}^m ?

This thesis keeps with tradition: Problem 1 is a foundational one here as well. The Roundness Theorem, the first of the two main results in this thesis is, roughly, a sufficient and (almost) necessary condition for a quadratic map to have a convex image. The Roundness Theorem is presented ahead in Section 2.2, where the meaning of an almost necessary condition is also explained. To the

best of my knowledge, the Roundness Theorem is the most general answer to Problem 1 that currently exists.

Problem 1 has subtleties. As the following examples demonstrate, the convexity of a quadratic map can be a delicate matter. The assertions made in each of the examples are justified in detail in Chapter 5 (see Examples 5.2.9 and 5.3.10).

Example 2.0.2. Let $\varepsilon \in \mathbb{R}$ and let $Q_\varepsilon : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the quadratic map given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} (x_1)^2 - (x_2)^2 \\ 2\varepsilon x_1 x_2 \\ (x_1)^2 + (x_2)^2 \end{pmatrix}$$

When $\varepsilon = 0$, the image of Q_ε is not convex, whereas When $\varepsilon \neq 0$, the image of Q_ε is convex.

When $\varepsilon = 0$, the quadratic map Q_ε can be thought of as a map from \mathbb{R}^2 into \mathbb{R}^3 . Ahead, Lemma 2.2.7 will explain that quadratic maps mapping from one space to another space of higher dimension are seldom convex. Lemma 2.2.7 plays an indispensable role in the proof of the Roundness Theorem; it is referred to throughout this thesis as the Instrumental Lemma.

Example 2.0.3. Let $\varepsilon \in \mathbb{R}$, and let $Q_\varepsilon : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the quadratic map given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 + \varepsilon x_1 x_2 \\ x_4^2 - \varepsilon x_1 x_2 \end{pmatrix}$$

When $\varepsilon = 0$, Q_ε is the standard quadratic map, whose image is convex. When $\varepsilon \neq 0$, the intersection of $\text{Im } Q_\varepsilon$ with the hyperplane $u^\perp \subset \mathbb{R}^4$, where $u = (0, 0, 1, 1)$ is not convex (this will be demonstrated in Chapter 4). Thus, when $\varepsilon \neq 0$, Q_ε is not convex.

Examples 2.0.2 and 2.0.3 capture the two primary means by which the image of a quadratic map Q can fail to be convex: either by having a codomain whose

dimension is larger than that of its domain, or by the existence of a hyperplane H for which the intersection $\text{Im } Q \cap H$ is not convex. These facts will crystallize with the development of the general theory of quadratic maps presented in Chapters 4 and 5.

Because it will have to be said so often, it will be useful to have a succinct way of saying that the image of a quadratic map is convex. In fact, there will soon be a need to say that the *restriction* of a quadratic map to a given subset of its domain has a convex image.

Definition 2.0.4. A quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **convex** if its image is a convex subset of \mathbb{R}^m . If $S \subset \mathbb{R}^n$, the restriction $Q|_S : S \rightarrow \mathbb{R}^m$ is **convex** if its image is convex.

Definition 2.0.4 may appear to carry with it some risk of confusion, for there is already a well known notion of convexity for maps that map from \mathbb{R}^n to \mathbb{R} (a map $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is usually defined as being convex if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$). However, this thesis will consider only those quadratic maps that map into \mathbb{R}^m for $m \geq 2$. The terminology of Definition 2.0.4 will always be unambiguous.

Finally, it is important to note that the convexification of the image of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be described in a simple way as a linear image of the cone of semidefinite $n \times n$ matrices; this is explored in Chapter 7. However, that description does little to address Problem 1, which inquires about the *difference* between the image of Q and the convexification thereof. Problem 1 will be solved using a different description of the convexification of the image of Q ; it is provided in Section 4.4,

2.1 A BRIEF HISTORY OF THE CONVEXITY OF QUADRATIC MAPS

This section provides an abridged history of Problem 1. Detailed histories can be found in the review papers [36] and [21]. Besides placing this thesis in its proper historical context, this chapter will also establish terminology and concepts that are needed for the statement of the first of this thesis' main result.

The history of Problem 1 consists of two threads that are rarely woven together. The first thread addresses Problem 1 directly and has led to a number of conditions, some necessary and some sufficient, for the convexity of quadratic maps mapping into low-dimensional spaces (\mathbb{R}^2 and \mathbb{R}^3 , specifically).

The second thread addresses a problem that, though superficially different from Problem 1, is equivalent to it in all cases of interest in this thesis. That problem has to do with the restriction of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ to the sphere \mathbb{S}^{n-1} , and whether or not the image of that restriction is a convex subset of \mathbb{R}^m . This line of questioning was put into motion by Toeplitz in his investigation of the *numerical range* of a complex matrix [41]. Toeplitz's study was brought to a conclusion by Hausdorff in [20], where he discovered what is now known as the Hausdorff-Toeplitz theorem [18].

The history of Problem 1 is short, for there have been few attempts to solve it. Moreover, the scope of those attempts was limited, either to quadratic maps mapping into low-dimensional spaces, \mathbb{R}^2 and \mathbb{R}^3 specifically (see Dines' and Calabi's convexity theorems, Theorems 2.1.1 and 2.1.5 below), or to quadratic maps possessing a very special form (see Theorem 2.1.8). Nonetheless, those attempts yielded important results, some of which influenced the early stages of this work.

The lineage of Problem 1 begins in the 1940s, with a paper by Dines [14]. In that paper, Dines proved the following theorem, and in doing so completely solved the quadratic convexity problem for quadratic maps that map into \mathbb{R}^2 .

Theorem 2.1.1 (Dines, [14]). *Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map. If $m = 2$, Q is convex.*

Dines proved his theorem by (almost) explicitly constructing, for each $x_1, x_2 \in \mathbb{R}^n$ and $t \in [0, 1]$, an $x \in \mathbb{R}^n$ for which $Q(x) = tQ(x_1) + (1 - t)Q(x_2)$. Dines' proof relies heavily on two-dimensional geometry. It does not lend itself to generalization. Theorem 2.1.1 will reappear ahead as a corollary to the general theory of quadratic convexity that is developed in Chapters 4 and 5 (see the end of Section 4.4).

Dines' theorem fails in higher dimensions. There exist nonconvex quadratic maps mapping into \mathbb{R}^m for every $m \geq 3$. For these maps, Problem 1 requires a means of distinguishing quadratic maps that are convex from those that are not. For quadratic maps that map into \mathbb{R}^3 , such a means was provided by Calabi in 1982 [10]. In the paper [10], Calabi provided a sufficient condition for a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^3$ to be convex, for $n \geq 3$. The statement of Calabi's condition requires the introduction of an important categorization of quadratic maps.

Definition 2.1.2. Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map defined by the matrices $M_1, \dots, M_m \in \text{Sym}_n$, and let $P \subset \text{Sym}_n$ be the linear span of the M_i . Q is **definite** if P contains a positive definite matrix, **semidefinite** if P contains a positive semidefinite matrix but no positive definite matrix, and **indefinite** if P contains no semidefinite matrices.

Example 2.1.3. The standard quadratic map is definite: the sum of the matrices that define the standard quadratic map from \mathbb{R}^n to itself is the $n \times n$ identity matrix.

Example 2.1.4. If, in example 2.0.2, $\varepsilon = 1$, then the matrix $M_3 \in \text{Sym}_3$ defining the third component of Q_ε is the identity map $I_3 \in \text{Sym}_3$; Q_ε is therefore definite. If $\varepsilon = -2$, then each of the matrices defining Q_ε is traceless and Q_ε is therefore indefinite. Finally, if $\varepsilon = 0$, Q_ε is semidefinite.

The significance of Definition 2.1.2 will be made clear in Chapter 4. In short, the geometry of the image of a quadratic map depends strongly on whether the map is definite, semidefinite or indefinite. If Q is definite, the image of Q , apart from the origin of \mathbb{R}^m , is contained in some open half-space of \mathbb{R}^m . If Q is semidefinite, the image of Q is contained in some closed half-space of \mathbb{R}^m , but not in any open half-space. And, finally, if Q is indefinite, the image of Q is not contained in any half-space of \mathbb{R}^m . This is all established in Chapter 4.

In this thesis, the emphasis will be squarely on definite quadratic maps. This is partly because the quadratically controlled control system that inspired this thesis (system (1.2.1)) is a definite quadratic map, but also because the analysis

of definite quadratic maps is more tractable than that of either semidefinite or indefinite maps.

For now, the notion of definiteness derives its importance from the following theorem, designated here as Calabi's Convexity Theorem.

Theorem 2.1.5 ((Calabi's Convexity Theorem) Calabi, [10]). *Let*

$Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map with $1 \leq m \leq 3$ and $n \geq 3$. If Q is definite, then it is convex.

Like Theorem 2.1.1, Theorem 2.1.5 cannot be generalized. There are nonconvex, *definite* quadratic maps mapping into \mathbb{R}^m for every $m \geq 4$. Example 2.0.3 offers a family of nonconvex quadratic maps from \mathbb{R}^4 to itself. The emphasis in this thesis will overwhelmingly be on *definite* quadratic maps that map into \mathbb{R}^m , with $m \geq 4$, and on the problem of distinguishing those that are convex from those that are not.

Until recently, Theorem 2.1.5 was the final word on Problem 1. Quadratic maps mapping into \mathbb{R}^m with $m \geq 4$ were left unexamined. Progress did eventually come, but in a form slightly different than that of Theorem 2.1.1 or Theorem 2.1.5. For, as the following subsection explains, Problem 1 comes in other guises.

PROBLEM 1 AND THE HAUSDORFF-TOEPLITZ THEOREM

In the course of studying Problem 1, a second problem arises. The convexity properties of some definite quadratic maps from \mathbb{R}^n to \mathbb{R}^m are determined by their restrictions to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. To be precise, if $M_m \in \text{Sym}_n$, the m^{th} matrix defining a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is equal to I_n , the identity map of \mathbb{R}^n , then the image of Q is convex if and only if its restriction to \mathbb{S}^{n-1} is convex (the proof of this statement is elementary, and will appear as an easy consequence of the theory developed in Chapter 4). For quadratic maps with the property that $M_m = I_n$, Problem 1 transforms into this,

Problem 1'. Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map defined by

$M_1, \dots, M_m \in \text{Sym}_n$, with $M_m = I_{\mathbb{R}^n}$. Let $\bar{Q} : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^m$ denote the restriction of Q to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. When is the image of \bar{Q} a convex subset of \mathbb{R}^m ?

Problem 1' is more general than it might appear at first. In fact, it bears on *all* definite quadratic maps, not just those satisfying the condition $M_m = I_n$. For, any given definite quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be made to satisfy the condition $M_m = I_n$ with the help of linear coordinate transformations. More precisely, if $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a definite quadratic map, then there exist invertible linear maps $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\tau : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that the composition $\tau \circ Q \circ \varphi$ is a quadratic map defined by M'_1, \dots, M'_m with $M'_m = I_n$ (the proof of this statement is also elementary, and is also deferred, this time to Section 4.6). Thus, *for definite quadratic maps, Problem 1' is equivalent to Problem 1.*

Problem 1' has a history of its own. It begins with the Hausdorff-Toeplitz theorem, an introduction to which can be found in the book [18]. To state the theorem, let $\langle, \rangle_{\mathbb{C}^n}$ denote the standard Hermitian form on \mathbb{C}^n , the standard n -dimensional complex space. The odd-dimensional sphere \mathbb{S}^{2n-1} can be thought of as the subset of \mathbb{C}^n consisting of those points z for which $\langle z, z \rangle_{\mathbb{C}^n} = 1$.

Theorem 2.1.6 (Hausdorff-Toeplitz). *Let H_1 and H_2 be complex, $n \times n$ Hermitian matrices ($n \geq 1$), and let $W : \mathbb{C}^n \rightarrow \mathbb{R}^2$ denote the map*

$$z \mapsto \begin{pmatrix} \langle z, H_1 z \rangle_{\mathbb{C}^n} \\ \langle z, H_2 z \rangle_{\mathbb{C}^n} \end{pmatrix}$$

Then, the restriction, $W|_{\mathbb{S}^{2n-1}}$ is convex.

\mathbb{C}^n , of course, can be identified with \mathbb{R}^{2n} by breaking each $z \in \mathbb{C}^n$ into its real and imaginary parts,

$$z \in \mathbb{C}^n \mapsto \begin{pmatrix} \operatorname{Re} z \\ \operatorname{Im} z \end{pmatrix} \in \mathbb{R}^{2n}$$

Similarly, the map

$$H \mapsto \begin{pmatrix} \operatorname{Re} H & \operatorname{Im} H \\ -\operatorname{Im} H & \operatorname{Re} H \end{pmatrix}$$

identifies the set of complex Hermitian $n \times n$ matrices with a certain subspace of Sym_{2n} , namely

$$\mathcal{C} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A \in \text{Sym}_n, B \in \mathfrak{so}(n) \right\} \subset \text{Sym}_{2n}$$

($\mathfrak{so}(n)$ is the space of real, $n \times n$ skew-symmetric matrices).

These identifications have the property that $\langle z, Hz \rangle_{\mathbb{C}^n} = \langle x, Mx \rangle_{\mathbb{R}^{2n}}$, where $x \in \mathbb{R}^{2n}$ is the element with which z is identified, and $M \in \text{Sym}_{2n}$ the symmetric matrix corresponding to the Hermitian matrix H . It follows that the map $W : \mathbb{C}^n \rightarrow \mathbb{R}^2$ can be understood as a quadratic map from \mathbb{R}^{2n} to \mathbb{R}^2 . Thus, the Hausdorff-Toeplitz theorem solves Problem 1' for the class of quadratic maps consisting of those that map from \mathbb{R}^{2n} to \mathbb{R}^2 , and whose two defining matrices lie in the subspace $\mathcal{C} \subset \text{Sym}_{2n}$ (for the remainder of this section, these quadratic maps will be referred to as *complex quadratic maps*).

There are a number of ways to generalize the quadratic map W that appears in Theorem 2.1.6. To begin with, the codomain of W need not be \mathbb{R}^2 ; it can be \mathbb{R}^m for any $m \geq 3$. The Hausdorff-Toeplitz theorem generalizes well when $m = 3$,

Theorem 2.1.7. *Let H_1, H_2 and H_3 be $n \times n$ Hermitian matrices with $n \geq 3$. Let $W : \mathbb{C}^n \rightarrow \mathbb{R}^3$ denote the map*

$$z \mapsto \begin{pmatrix} \langle z, H_1 z \rangle_{\mathbb{C}^n} \\ \langle z, H_2 z \rangle_{\mathbb{C}^n} \\ \langle z, H_3 z \rangle_{\mathbb{C}^n} \end{pmatrix}$$

Then, the restriction, $W|_{\mathbb{S}^{2n-1}}$ is convex.

Theorem 2.1.7 has appeared in numerous places, [4] and [1], for example.

Complex quadratic maps that map into \mathbb{R}^m for $m \geq 4$ are not unconditionally convex; nonconvex examples can be found in [17]. That paper, incidentally, completely solves Problem 1' for complex quadratic maps. It provides a sufficient and (almost) necessary condition for an arbitrary complex quadratic map to be

convex. The condition given in [17] is the most general condition for the convexity of a complex quadratic map to have been discovered so far; it is stated below as Theorem 2.1.8, without further explanation. Section 7.5 will discuss the proof of Theorem 2.1.8 and its relationship to this thesis' own convexity result, the Roundness Theorem. For now, it will only be said that the condition in [17] is far less general than the Roundness Theorem, and that the proof of the latter is simpler than that of the former.

Theorem 2.1.8. *Let H_1, \dots, H_m be $n \times n$ Hermitian matrices and let $W : \mathbb{C}^n \rightarrow \mathbb{R}^m$ be the complex quadratic map defined by the H_i . Suppose that the following conditions hold,*

- *The multiplicity of the largest eigenvalue of $\sum_{i=1}^m \lambda_i H_i$ is independent of $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$.*
- *If $E_1(\lambda)$ denotes the eigenspace corresponding to the largest eigenvalue of $\sum_{i=1}^m \lambda_i H_i$, where $\lambda = (\lambda_1, \dots, \lambda_m)$, then the union $\bigcup_{\lambda \in \mathbb{R}^m} E_1(\lambda)$ is not equal to all of \mathbb{C}^n .*

Then, the restriction $W|_{\mathbb{S}^{2n-1}}$ is convex.

BRICKMAN'S THEOREM

The Hausdorff-Toeplitz theorem is a special case of a theorem of Brickman's, [6].

Theorem 2.1.9 (Brickman, [6]). *Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^2$ be a quadratic map. If $n \geq 3$, then the restriction $Q|_{\mathbb{S}^{n-1}}$ is convex.*

Above, the Hausdorff-Toeplitz theorem was reinterpreted as saying that certain quadratic maps from \mathbb{R}^{2n} to \mathbb{R}^2 , those whose defining matrices lie in the subspace $\mathcal{C} \subset \text{Sym}_{2n}$, are convex when restricted to \mathbb{S}^{2n-1} . Brickman's theorem says that *every* quadratic map mapping into \mathbb{R}^2 from \mathbb{R}^n , with n odd or even, is convex when restricted to \mathbb{S}^{n-1} .

Brickman's theorem completely solves Problem 1' for quadratic maps that map into \mathbb{R}^2 . Sadly, it defies generalization. For every $m \geq 3$, there exist quadratic

maps $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for which the restriction $Q|_{\mathbb{S}^{n-1}}$ is not convex; the Hopf map provides a particularly famous example.

Example 2.1.10. Consider the quadratic map $Q_H : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by the matrices

$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The restriction $Q_H|_{\mathbb{S}^3}$ is the **Hopf map**, a well-known map that maps S^3 surjectively into S^2 . Thus, the image of $Q_H|_{\mathbb{S}^3}$, namely \mathbb{S}^2 , is not convex.

Chapter 5 will construct more pedestrian examples of quadratic maps $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose restrictions to \mathbb{S}^{n-1} are not convex.

2.2 THIS THESIS' FIRST CONTRIBUTION: THE ROUNDNESS THEOREM

Calabi's theorem (Theorem 2.1.5) can be strengthened. For the class of quadratic maps specified in his theorem (those mapping from \mathbb{R}^n to \mathbb{R}^m with $1 \leq m \leq 3$ and $n \geq 3$), definiteness is a guarantor not only of convexity, but of *stable convexity* as well.

Definition 2.2.1. A quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $M_1, \dots, M_m \in \text{Sym}_n$ is **stably convex** if it is convex and if there exists an open neighbourhood $U \subset \text{Sym}_n \times \dots \times \text{Sym}_n$ of (M_1, \dots, M_m) with the following property: for every $(M'_1, \dots, M'_m) \in U$, the quadratic map Q' defined by M'_1, \dots, M'_m is also convex.

Informally, a quadratic map defined by the matrices $M_1, \dots, M_m \in \text{Sym}_n$ is stably convex if it is convex *and* if it remains convex under sufficiently small perturbations of the M_i .

For the class of maps specified in Calabi's theorem, definiteness is a sufficient condition for stable convexity, in addition to plain old convexity, because definiteness itself is preserved under sufficiently small perturbations: if the linear span of some $M_1, \dots, M_m \in \text{Sym}_n$ contains a positive definite matrix, so too does that of any $M'_1, \dots, M'_m \in \text{Sym}_n$ that are sufficiently close to the M_1, \dots, M_m . Calabi's Convexity Theorem, Theorem 2.1.5, can therefore be restated as follows.

Theorem 2.2.2 ((Calabi's Stable Convexity Theorem)). Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map with $n \geq 3$ and $1 \leq m \leq 3$. If Q is definite, then it is stably convex.

Unfortunately, Theorem 2.2.2 applies only to quadratic maps that map into \mathbb{R} , \mathbb{R}^2 or \mathbb{R}^3 . Example 2.0.3 shows that definiteness is *not* a sufficient condition for the stable convexity of quadratic maps that map into \mathbb{R}^m for any $m \geq 4$.

Prior to this thesis, nothing was known about the stable convexity of definite quadratic maps mapping into \mathbb{R}^m for $m \geq 4$. The first half of this thesis establishes a necessary and sufficient condition for the stable convexity of quadratic maps mapping from \mathbb{R}^n to \mathbb{R}^m with $n \geq m, m \geq 4$. The condition is that of *roundness*,

Definition 2.2.3. A quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $M_1, \dots, M_m \in \text{Sym}_n$ is **round** if it has the following two properties:

- Q is definite (i.e. some linear combination of the M_i is positive definite).
- Every linear combination of the M_i that is (i) nonzero, (ii) degenerate (i.e., not full rank), and (iii) semidefinite has rank $n - 1$.

Round quadratic maps will be given a more succinct definition in Chapter 4. The geometric significance of roundness will be explained in a moment (see Remark 2.2.6 below); the existence of round quadratic maps is addressed in Section 4.5 of Chapter 4.

In the terms of Definition 2.2.3, this thesis's first principal result is this,

Theorem 2.2.4. (*The Roundness Theorem*) *Let n, m be natural numbers with $m \geq 4, n \geq m, n \neq m + 1$. If $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a definite quadratic map then Q is stably convex if and only if it is round.*

The condition $n \neq m + 1$ will be explained in Chapter 5. Theorem 2.2.4 says that if a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $M_1, \dots, M_m \in \text{Sym}_n$ (with $n \neq m + 1$) is round, then it is convex, and its convexity is immune to small changes in the M_i . If, on the other hand, Q is not round, then Q is either not convex or can be made nonconvex by arbitrarily small changes to the M_i that define it.

Example 2.2.5. *When $n \geq 3$, the standard quadratic map is not round. The span of the $M_i \in \text{Sym}_n$ that define the standard quadratic map contains each of the $E_{i,i}$ defined in example 2.0.1. The $E_{i,i}$ are strictly positive semidefinite but have rank 1. Examples 2.0.3 shows that the standard quadratic map in four dimensions, which is convex, can be made nonconvex by arbitrarily small perturbations to the $M_i \in \text{Sym}$ that define it.*

Remark 2.2.6. Section 4.5 in Chapter 4 will show that when $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies the condition of Definition 2.2.3, the boundary of the image of Q (apart from the origin $o \in \mathbb{R}^m$) is a smoothly embedded submanifold of \mathbb{R}^m . This is in contrast to nonround quadratic maps, like the standard one, whose boundaries can have corners. This is reason for the name *round*. ■

Section 4.5 will show that a round quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has the following property: the boundary of $\text{Conv}(\text{Im } Q)$, (the convexification of the image of Q) is nonempty and is contained in the image $\text{Im } Q$ itself. Thus, a round quadratic map can fail to be convex only if there is a point in the *interior* of

$\text{Conv}(\text{Im } Q)$ that is not contained in $\text{Im } Q$. Now, when Q is round, the image of Q is a closed subset of \mathbb{R}^m (see Corollary 4.4.5 in Chapter 4). Thus, if there is a point in the interior of $\text{Conv}(\text{Im } Q)$ that is not in $\text{Im } Q$ itself, it follows that there is a whole open set of points in the interior of $\text{Conv}(\text{Im } Q)$ that are also not contained in $\text{Im } Q$. Thus, if Q is round, the only way that it can fail to be convex is if its image has a *bubble*, an open set contained in the interior of $\text{Conv}(\text{Im } Q)$ that is not contained in $\text{Im } Q$.

Half of the proof of the backward implication in Theorem 2.2.4 (roundness \Rightarrow stably convex) refutes the existence of bubbles in $\text{Im } Q$ when Q is round. The remaining half shows that roundness is a *stable* property: if $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$, defined by the matrices $M_1, \dots, M_m \in \text{Sym}_n$ is round, so too is every quadratic map $Q' : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by matrices $M'_1, \dots, M'_m \in \text{Sym}_n$ that are sufficiently close to those that define Q .

If Q is not round, in every neighbourhood of $(M_1, \dots, M_m) \in \text{Sym}_n$ there exists a tuplet $(M'_1, \dots, M'_m) \in \text{Sym}_n$ that defines a quadratic map Q' with the following property: there exists a point in the *boundary* of $\text{Conv}(\text{Im } Q')$ that is not contained in $\text{Im } Q'$. It follows that Q is not stably convex, and that Q can be made nonconvex by perturbing the boundary of its image. The demonstration of this fact will complete the proof of Theorem 2.2.4.

Finally, in the course of proving Theorem 2.2.4 an important lemma emerges,

Lemma 2.2.7. (*The Instrumental Lemma*) *Suppose that $m > n$ are natural numbers. Then, no definite quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is stably convex.*

Lemma 2.2.7 plays an important role in the proof of the forward implication (stably convex \Rightarrow roundness) in Theorem 2.2.4.

3

Mathematical Preliminaries

3.1 INTRODUCTION

This chapter establishes the mathematical background required for Chapters 4 and 5.

Sections 3.3 through 3.4.5 state some basic facts about convex sets and cones that will be used throughout the Chapters 4 and 5. These sections culminate in a useful description of the convexification $\text{Conv}(K)$ of a cone $K \subset \mathbb{R}^n$. In short, under mild conditions on K , the convexification of K is equal to its *double dual* K^{**} (to be defined ahead, in Section 3.4.3). This description of $\text{Conv}(K)$ will turn out to be particularly suited to the study of quadratic maps.

Section 3.4.6 studies systems of homogeneous, multivariate quadratic equations. These play an important role in calculating the preimage of a point under a quadratic map.

Section 3.6 establishes a few basic facts about the compact-open topology. These will be used to study the topology of various subsets of the set of quadratic maps from one space to another. The results of this section will, among other things, help to explain why roundness is an open, or stable, condition.

Section 3.7 proves a simple fact about homogeneous maps between vector spaces (maps $f : V \rightarrow W$ satisfying for which there exists a $k \in \mathbb{Z}$ satisfying $f(\lambda x) = \lambda^k f(x)$, for all $x \in V, \lambda \in \mathbb{R}$). This fact will provide conditions under which the image of a quadratic map is closed.

Finally, Section 3.8 covers a small variety of topics from differential geometry. The material covered in this section is used exclusively in the proof of sufficiency in the Roundness Theorem.

3.2 NOTATION

This section establishes all of the notation that is used throughout Part I of this thesis. Additional notation will be introduced as it is needed.

Numbers. \mathbb{Z}^+ and $\mathring{\mathbb{Z}}^+$ will respectively denote the nonnegative and positive integers, while \mathbb{R}^+ and $\mathring{\mathbb{R}}^+$ will denote the nonnegative and positive real numbers.

Standard Bases. Members of the standard basis of \mathbb{R}^n will be written as e_i , $i = 1, \dots, n$.

Subsets. If X is a topological space and $A \subset X$, $\text{Int}_X(A)$, $\text{Cl}_X A$ and $\partial_X A$ will respectively denote the interior, closure and boundary of A in X . The complement of A in X will be denoted by A^c .

Restrictions. If $F : X \rightarrow Y$ is a map from some space X to another space Y , then the restriction of F to a subset $A \subset X$ (that is, the composition $F \circ i_A$ where $i_A : A \hookrightarrow X$ is the inclusion of A into X) will be written as $F|_A : A \rightarrow Y$.

Inner product spaces. All vector spaces in this thesis are finite-dimensional. Inner products and their associated norms will be written respectively as \langle, \rangle and

$\|\cdot\|$. When necessary, inner products and their norms will carry a subscript denoting the spaces that carries them, such as $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$.

Subsets of inner product spaces. If V is an inner product space, the unit sphere in V , $\{x \in V \mid \|x\|_V = 1\}$ will be written as \mathbb{S}^V . If $V = \mathbb{R}^n$, the unit sphere will be written as usual as \mathbb{S}^{n-1} . \mathbb{S}^{n-1} will sometimes be thought of as a subset of the unit disk $\mathbb{D}^n = \{x \in \mathbb{R}^n : \|x\|_{\mathbb{R}^n} \leq 1\}$.

If $A \subset V$ is a subset then A^\perp will denote the orthogonal complement of A . If A happens to be a singleton subset, $A = \{u\}$ with $u \in V$, then A^\perp will be written as u^\perp .

Halfspaces. If V is again an inner product space, three additional subsets will be associated with each $u \in V$ (and each $a \in \mathbb{R}$) :

$$\begin{aligned} H_u^a &= \{x \in V \mid \langle x, u \rangle = a\} \\ H_u^+ &= \{x \in V \mid \langle x, u \rangle \geq 0\} \\ \overset{\circ}{H}_u^+ &= \{x \in V \mid \langle x, u \rangle > 0\} \end{aligned}$$

Linear maps. If V and W are both vector spaces, $\mathcal{L}(V, W)$ will denote the set of linear maps from V to W . The identity map on a space V will be written as I_V unless $V = \mathbb{R}^n$ in which case it will be written as I_n .

Elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ will sometimes be identified with their matrices, relative to the standard bases of \mathbb{R}^n and \mathbb{R}^m , in which case the $(i, j)^{th}$ element of $M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ will be written as M_{ij} .

Symmetric maps If V is an inner product space, $\text{Sym}(V) \subset \mathcal{L}(V, V)$ will denote the set of maps that are symmetric with respect to the inner product on V :

$$\text{Sym}(V) = \{M \in \mathcal{L}(V, V) \mid \langle Mx, y \rangle = \langle x, My \rangle \ \forall \ x, y \in V\}$$

$\text{Sym}(V)$ carries the inner product $\langle M_1, M_2 \rangle_{\text{Sym}(V)} = \text{Trace}(M_1 M_2)$, where the trace is induced by the inner product on V .

If $L \subset V$ is a subspace, $\text{Sym}(L)$ will be understood as the set of linear maps from L to itself that are symmetric with respect to the inner product on L that is induced by the one on V .

Positive semidefinite maps. $S^+(V) \subset \text{Sym}(V)$ will denote the set of positive semidefinite maps,

$$S^+(V) = \{M \in \text{Sym}(V) : \langle x, Mx \rangle_V \geq 0 \forall x \in V\}$$

while $\mathring{S}^+(V) \subset \text{Sym}(V)$ will denote the set of positive definite symmetric ones,

$$\mathring{S}^+(V) = \{M \in \text{Sym}(V) : \langle x, Mx \rangle_V > 0 \forall x \in V\}$$

The set difference $S^+(V) - \mathring{S}^+(V)$ will be written as $\partial S^+(V)$; this is the set of positive semidefinite symmetric maps that are not positive definite. If V happens to be \mathbb{R}^n for some n , the foregoing sets will be written as $\text{Sym}_n, S_n^+, \mathring{S}_n^+$ and ∂S_n^+ . The subscript $^\circ$ is meant to conjure the oft-used notation \mathring{A} for the interior of a subset $A \subset X$ of a topological space X (even though that notation is not used here). After all, $\mathring{S}^+(V)$ is the interior of $S^+(V)$ relative to $\text{Sym}(V)$.

Rank 1 positive semidefinite maps. There is a map from V to $\text{Sym}(V)$ taking $v \in V$ to the map $w \mapsto \langle v, w \rangle_V v$ (or, more prosaically in the case of Sym_n, vv^T). This map will be denoted by $q : V \rightarrow \text{Sym}(V)$. The image of $v \in V$ under q will be written as $v \otimes v$ (unless $V = \mathbb{R}^n$, in which case $v \otimes v$ will be denoted as usual by vv^T).

3.3 CONVEX SETS

This section cobbles together some basic facts about convex sets in \mathbb{R}^n . Most of the material in this section is elementary and presented without proof. For a reference, see [2].

Definition 3.3.1. A subset $A \subset \mathbb{R}^n$ is **convex** if for all $t \in [0, 1]$ and $x, y \in A$, $tx + (1 - t)y$ is also in A .

Remark 3.3.2. A basic but important test for the convexity of a subset $A \subset \mathbb{R}^n$ follows immediately from Definition 3.3.1: $A \subset \mathbb{R}^n$ is convex if and only if $A \cap H_u^a$ is convex for all affine hyperplanes $H_u^a \subset \mathbb{R}^n$. This test will be used in the next chapter; it will turn out that for any quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that is not round, there exists a linear hyperplane $p^\perp \subset \mathbb{R}^m$ for which $\text{Im } Q \cap p^\perp$ is either not convex, or can be made nonconvex by an arbitrarily small perturbation to Q .

The intersection of a collection of convex subsets of \mathbb{R}^n is itself a convex subset. It follows that there exists a smallest convex subset containing a given subset $A \subset \mathbb{R}^n$, namely the intersection of all the convex subsets of \mathbb{R}^n that contain A .

Definition 3.3.3. The **convexification** of a subset $A \subset \mathbb{R}^n$ is the smallest convex subset of \mathbb{R}^n containing A . It is denoted by $\text{Conv}(A)$.

Remark 3.3.4. It follows from Definition 3.3.3 that A is convex if and only if $A = \text{Conv}(A)$. Despite its simplicity, this observation will prove useful; it provides a test for the convexity of $A \subset \mathbb{R}^n$ in the event that $\text{Conv}(A)$ is known and the containments $A \subset \text{Conv}(A)$ and $\text{Conv}(A) \subset A$ readily verifiable. This test will be applied to quadratic maps in the next chapter.

$\text{Conv}(A)$ has a more practical description than the one provided in Definition 3.3.3,

$$\text{Conv}(A) = \left\{ \sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{Z}^+, a_i \in A, \lambda_i \in \mathbb{R}^+, \sum_{i=1}^m \lambda_i = 1 \right\} \quad (3.3.1)$$

Note that (3.3.1) implies $\text{Conv}(L(A)) = L(\text{Conv}(A))$ for any linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and any $A \subset \mathbb{R}^n$.

The following collection of lemmas describes some basic properties of convex sets and of convexifications. Proofs can be found in [2].

Lemma 3.3.5. Let $A \subset \mathbb{R}^n$.

- (i) If A is compact, so is $\text{Conv}(A)$.
- (ii) If A is convex then $\text{Cl}_{\mathbb{R}^n}(A)$ is convex.

- (iii) Suppose that A is closed and convex. If $\text{Int}_{\mathbb{R}^n}(A) \neq \emptyset$, then $A = \text{Cl}_{\mathbb{R}^n}(\text{Int}_{\mathbb{R}^n}(A))$.
- (iv) If A is convex and $\text{Int}_{\mathbb{R}^n}(A) \neq \emptyset$ then $\text{Int}_{\mathbb{R}^n}(A)$ is convex.
- (v) If A is convex and $\text{Int}_{\mathbb{R}^n}(A) = \emptyset$ then there exists a proper affine subspace $P \subset \mathbb{R}^n$ such that $A \subset P$.
- (vi) If A is convex and dense then $A = \mathbb{R}^n$.

3.4 CONES

Definition 3.4.1. A subset $K \subset \mathbb{R}^n$ is a **cone** if for all $\lambda \in \mathbb{R}^+$ and $k \in K$, λk is also in K .

This section assembles together a variety of facts about cones in \mathbb{R}^n . Throughout this section \mathbb{R}^n carries its standard inner product. The results of this section apply to cones in any finite-dimensional real inner product space.

Cones are of central importance in this thesis simply because the image of every quadratic map is a cone. By Remark 3.3.4, a quadratic map Q is convex if and only if $\text{Im}(Q) = \text{Conv}(\text{Im}(Q))$. This section, therefore, focuses not only on cones but on their convexifications as well. One basic fact about the convexification of a cone is that it too is a cone.

Proposition 3.4.2. If $K \subset \mathbb{R}^n$ is a cone, so are $\text{Conv}(K)$ and $\text{Cl}_{\mathbb{R}^n}(K)$.

Proof. That $\text{Conv}(K)$ is a cone follows from equation (3.3.1). To see that $\text{Cl}_{\mathbb{R}^n}(K)$ is cone, suppose that $k \in \text{Cl}_{\mathbb{R}^n}(K)$, $\lambda \in \mathbb{R}^+$ and that $U \subset \mathbb{R}^n$ is a neighbourhood of λk in \mathbb{R}^n . $\frac{1}{\lambda}U$ is an open neighbourhood of k . $U \cap K$ therefore contains $\lambda k'$, where k' is any element of $\frac{1}{\lambda}U \cap K$. It follows that $\lambda k \in \text{Cl}_{\mathbb{R}^n}(K)$. \square

As was the case with convex sets, the intersection of a collection of cones in \mathbb{R}^n is again a cone. This allows for the definition of the smallest cone containing a given subset $A \subset \mathbb{R}^n$, namely as the intersection of all the cones in \mathbb{R}^n that contain A .

Defintion 3.4.3. Let $A \subset \mathbb{R}^n$. The **cone generated by A** is the smallest cone containing A . It is denoted by $\text{Cone}(A)$.

Like $\text{Conv}(A)$, $\text{Cone}(A)$ has a less abstract representation than the one given in Definition 3.4.3,

$$\text{Cone}(A) = \{\lambda a \mid \lambda \in \mathbb{R}^+, a \in A\} \quad (3.4.1)$$

There is similarly a smallest *convex* cone containing a given subset $A \subset \mathbb{R}^n$. This, predictably now, is the intersection of all convex cones containing A .

Defintion 3.4.4. Let $A \subset \mathbb{R}^n$. The **convex cone generated by A** is the smallest convex cone containing A .

Proposition 3.4.5. Let $A \subset \mathbb{R}^n$.

- (i) $\text{Cone}(\text{Conv}(A)) = \text{Conv}(\text{Cone}(A))$
- (ii) The convex cone generated by A is equal to $\text{Cone}(\text{Conv}(A))$ (which, by (i), is equal to $\text{Conv}(\text{Cone}(A))$).

Proof. Every element in $\text{Conv}(\text{Cone}(A))$ can be written as $\sum_{i=1}^m \gamma_i \lambda_i a_i$ where $\lambda_i \in \mathbb{R}^+$, $a_i \in A$ and $\sum_{i=1}^m \gamma_i = 1$. But,

$$\sum_{i=1}^m \gamma_i \lambda_i a_i = \beta \cdot \sum_{i=1}^m \frac{\gamma_i \lambda_i}{\beta} a_i \quad (3.4.2)$$

where $\beta = \sum_{i=1}^m \gamma_i \lambda_i$. The right side of (3.4.2) is an element of $\text{Cone}(\text{Conv}(A))$. Thus, $\text{Conv}(\text{Cone}(A)) \subset \text{Cone}(\text{Conv}(A))$. The opposite inclusion follows similarly from the observation that every element of $\text{Cone}(\text{Conv}(A))$ can be written as $\lambda \cdot \sum_{i=1}^m \gamma_i a_i$ with $\lambda \in \mathbb{R}^+$, $a_i \in A$ and $\sum_{i=1}^m \gamma_i = 1$. This proves (i).

The equality $\text{Cone}(\text{Conv}(A)) = \text{Conv}(\text{Cone}(A))$ implies that $\text{Cone}(\text{Conv}(A))$ is a convex cone. It therefore contains the convex cone generated by A . At the same time, the convex cone generated by A is a cone that contains $\text{Conv}(A)$. It must therefore contain $\text{Cone}(\text{Conv}(A))$. This proves (ii). \square

It follows from Proposition 3.4.5 that if $A = K$ is a cone, then the convex cone generated by K is just $\text{Conv}(K)$.

Finally, it was observed earlier (part (v) of Lemma 3.3.5) that a convex set in \mathbb{R}^n whose interior is empty is contained in a proper affine subspace of \mathbb{R}^n . There is a useful variant of this statement that applies specifically to cones.

Proposition 3.4.6. *Let $K \subset \mathbb{R}^n$ be a convex cone. If $\text{Int}_{\mathbb{R}^n}(K) = \emptyset$ then there exists a proper linear subspace $P \subset \mathbb{R}^n$ such that $K \subset P$.*

Proof. Lemma 3.3.5 implies that there exists a proper affine subspace $P \subset \mathbb{R}^n$ satisfying $K \subset P$. But, $0 \in K$ implies that P is a linear subspace. \square

3.4.1 BASES

A convenient way to study cones is through their bases.

Definition 3.4.7. *Let $K \subset \mathbb{R}^n$ be a cone. A subset $B_K \subset K$ is a **base** for K if for every nonzero $k \in K$ there exists a unique $\lambda \in \mathbb{R}^+$ and a unique $b \in B_K$ such that $k = \lambda b$.*

It follows from Definition 3.4.7 that if B_K is a base for the cone $K \subset \mathbb{R}^n$, then $0 \notin B_K$, and $K = \text{Cone}(B_K)$.

Bases are useful because they impart information about the cones that they generate. The next proposition demonstrates a pertinent example of this. More examples will follow.

Proposition 3.4.8. *Let $K \subset \mathbb{R}^n$ be a cone and let B_K be a base for K . Then, $\text{Conv}(K) = \text{Cone}(\text{Conv}(B_K))$.*

Proof. Proposition 3.4.5 implies that $\text{Cone}(\text{Conv}(B_K)) = \text{Conv}(\text{Cone}(B_K))$. But, $\text{Conv}(\text{Cone}(B_K)) = \text{Conv}(K)$, since B_K is a base for K . \square

Corollary 3.4.9. *Let $K \subset \mathbb{R}^n$ be a cone. If K has a convex base then K is convex.*

Proof. If B_K is a convex base for K , then $B_K = \text{Conv}(B_K)$. By the lemma, $\text{Conv}(K) = \text{Cone}(\text{Conv}(B_K)) = \text{Cone}(B_K) = K$. It follows that K is convex. \square

Two types of bases will play prominent roles in the study of quadratic maps: spherical bases, and flat bases.

SPHERICAL BASES

If $K \subset \mathbb{R}^n$ is a cone, $K \cap \mathbb{S}^{n-1}$ is a base for K .

Defintion 3.4.10. Let $K \subset \mathbb{R}^n$ be a cone. The base $K \cap \mathbb{S}^{n-1}$ is the **spherical base** for K . It is denoted by B_K^{sph} .

Remark 3.4.11. Note that $B_K^{\text{sph}} = \emptyset$ if and only if $K = \mathbf{o}$. ■

Let $K \subset \mathbb{R}^n$ be a cone and let B_K be any base for K . Because $\mathbf{o} \in \mathbb{R}^n$ is not contained in B_K , the spherical base for K can be obtained by radially projecting B_K onto \mathbb{S}^{n-1} : $B_K^{\text{sph}} = r_n(B_K)$, where $r_n : \mathbb{R}^n - \{\mathbf{o}\} \rightarrow \mathbb{S}^{n-1}$ is the radial retraction $x \mapsto x/||x||$. In fact, as the next Proposition shows, B_K^{sph} can be obtained in this way from any subset of $\mathbb{R}^n - \{\mathbf{o}\}$ that generates K .

Proposition 3.4.12. Let $A \subset \mathbb{R}^n - \{\mathbf{o}\}$. If $K = \text{Cone}(A)$ then $B_K^{\text{sph}} = r_n(A)$, where $r_n : \mathbb{R}^n - \{\mathbf{o}\} \rightarrow \mathbb{S}^{n-1}$ is the radial retraction $x \mapsto x/||x||$.

The closedness of the image of a quadratic map will turn out to be an important property. The following proposition says that the closedness of a cone is encoded in its spherical base.

Proposition 3.4.13. Let $K \subset \mathbb{R}^n$ be a cone. K is closed if and only if B_K^{sph} is closed (and therefore compact).

Proof. If K is closed, $B_K^{\text{sph}} = K \cap \mathbb{S}^{n-1}$ is the intersection of two closed subsets and therefore closed itself. Conversely, suppose that B_K^{sph} is closed and consider a sequence $k_n \in K$ that converges to a point $k \in \mathbb{R}^n$. If $k = \mathbf{o}$, then $k \in K$. If $k \neq \mathbf{o}$, then $k_n \neq \mathbf{o}$ for all $n \geq n_o$, for some $n_o \in \mathbb{Z}^+$. The sequence $k_n/||k_n|| \in B_K^{\text{sph}}$ converges to $k/||k||$, which, by the closedness of B_K^{sph} , must itself be in B_K^{sph} . It follows that $k \in K$, since $k = ||k|| \cdot k/||k||$. K is therefore sequentially closed. □

FLAT BASES

The next proposition describes another useful way to obtain bases for cones.

Proposition 3.4.14. *Let $K \subset \mathbb{R}^n$ be a cone and suppose that $u \in \mathbb{R}^n$ satisfies $\langle u, k \rangle > 0$ for all nonzero $k \in K$. Then, for any $\alpha > 0$, $H_u^\alpha \cap K$ is a base for K .*

Proof. Every nonzero $k \in K$ can be written uniquely as $k = \lambda h$ with $\lambda > 0$ and $h \in H_u^\alpha \cap K$ by setting $\lambda = \langle k, u \rangle / \alpha$ and $h = \alpha / \langle k, u \rangle \cdot k$. \square

Defintion 3.4.15. *Let $K \subset \mathbb{R}^n$ be a cone. A base for K having the form $H \cap K$ where $H \subset \mathbb{R}^n$ is an affine hyperplane is a **flat base** for K .*

Remark 3.4.16. Note that if $B_K^H = K \cap H$ is a flat base for the cone K , then $H = H_u^\alpha$ for some $\alpha > 0$ and some $u \in \mathbb{R}^n$ such that $\langle u, k \rangle > 0$ for all nonzero $k \in K$. It follows that if k_1 and k_2 are distinct, and both lie in a flat base for K , they are necessarily linearly independent (if $k_2 = \lambda k_1$, then $\langle u, k_2 \rangle = \alpha = \lambda \langle u, k_1 \rangle = \lambda \alpha$, and $\lambda = 1$).

Also, if $B \subset H_u^\alpha$ is nonempty, where $u \neq 0$ and $\alpha > 0$, then B is a flat base for $\text{Cone}(B)$. \blacksquare

Flat bases will be used in the proof of the Roundness Theorem. The next proposition in particular will be used to test the convexity of quadratic maps.

Proposition 3.4.17. *Let $K \subset \mathbb{R}^n$ be a cone. K is convex if and only if every flat base of K is convex.*

Proof. If K is convex, $K \cap H$ is convex for any affine hyperplane $H \subset \mathbb{R}^n$. If K has a flat base that is convex, K is convex by Corollary 3.4.9. \square

The next proposition says that the closedness of a cone $K \subset \mathbb{R}^n$ can be detected by flat bases just as well as it can by spherical ones.

Proposition 3.4.18. *Let $K \subset \mathbb{R}^n$ be a cone and let $B_K = K \cap H_u^\alpha$ be a flat base for K with $u \in \mathbb{R}^n$ and $\alpha > 0$. Then, K is closed if and only if B_K is compact.*

Proof. This follows from Lemma 3.4.13: if B_K is compact then so is B_K^{sph} , since by Lemma 3.4.12, $B_K^{\text{sph}} = r_n(B_K)$. Conversely, if K is closed, then B_K^{sph} is compact. But then so too is B_K , for $B_K = f(B_K^{\text{sph}})$ where $f: \mathbb{S}^{n-1} \cap \mathring{H}_u^+ \rightarrow H_u^\alpha$ is the continuous map $x \mapsto \alpha / \langle x, u \rangle \cdot x$. \square

The final result of this section establishes a simple relationship between the interior of a given cone and the interior of a flat base of that same cone.

Proposition 3.4.19. *Let $K \subset \mathbb{R}^n$ be a cone and let $B_K = K \cap H_u^a$ be a flat base for K . Then, $\text{Int}_{H_u^a}(B_K) = H_u^a \cap \text{Int}_{\mathbb{R}^n}(K)$.*

Proposition 3.4.19 follows immediately from the next two lemmas. Note that Proposition 3.4.19 implies that the boundary of K is equal to $\text{Cone}(\partial_{H_u^a} B_K)$.

Lemma 3.4.20. *Let $u \in \mathbb{R}^n$ be nonzero and $a > 0$. Then, the map*

$$H_u^a \times \mathring{\mathbb{R}}^+ \rightarrow \mathring{H}_u^+ \text{ given by } (x, \lambda) \mapsto \lambda x \text{ is a homeomorphism with inverse } x \mapsto \left(\frac{a}{\langle x, u \rangle} x, \frac{\langle x, u \rangle}{a} \right).$$

Lemma 3.4.21. *Let $K \subset \mathbb{R}^n$ be a cone and let $B_K = K \cap H_u^a$ be a flat base for K . Then, $\text{Int}_{\mathbb{R}^n}(K) = \text{Cone}(\text{Int}_{H_u^a}(B_K)) - \{0\}$.*

Proof. Suppose that $p \in \text{Cone}(\text{Int}_{H_u^a}(B_K)) - \{0\}$. Then, $p' = a/\langle p, u \rangle \cdot p$ lies in the interior of B_K relative to H_u^a . Let $U \subset H_u^a$ be an open neighbourhood of p' in H_u^a that is contained in B_K . Then, the image of $U \times \mathring{\mathbb{R}}^+$ under the homeomorphism described in Lemma 3.4.20 is an open neighbourhood of p in \mathbb{R}^n that is contained in K . Thus, $\text{Cone}(\text{Int}_{H_u^a}(B_K)) - \{0\} \subset \text{Int}_{\mathbb{R}^n}(K)$.

Conversely, if $p \in \text{Int}_{\mathbb{R}^n}(K)$, let $U \subset \mathbb{R}^n$ be an open neighbourhood of p that is contained in K . The image of U under the inverse of the homeomorphism in the proposition is an open neighbourhood in $H_u^a \times \mathring{\mathbb{R}}^+$ and therefore contains a subset of the form $U' \times I$, where $U' \subset H_u^a$ is an open neighbourhood of $a/\langle p, u \rangle$ in $H_u^a \times \mathring{\mathbb{R}}^+$ and $I \subset \mathring{\mathbb{R}}^+$ an open set. It follows that p is in $\text{Cone}(\text{Int}_{H_u^a}(B_K))$.

■.

□

3.4.2 DUAL CONES

One important way to construct cones is through dualization,

Definition 3.4.22. *Let $A \subset \mathbb{R}^n$. The **dual cone** of A is the subset*

$$\begin{aligned} A^* &= \{u \in \mathbb{R}^n \mid \langle u, a \rangle \geq 0 \ \forall \ a \in A\} \\ &= \{u \in \mathbb{R}^n \mid A \subset H_u^+\} \end{aligned}$$

The dual cone of A will be denoted by A^* .

Remark 3.4.23. The dual cone of any subset $A \subset \mathbb{R}^n$ is indeed a cone, and a closed, convex one at that. The closedness and convexity of A^* follow respectively from the continuity and bilinearity of the inner product on \mathbb{R}^n . ■

In the next chapter, quadratic maps will be studied primarily through the dual cones of their images. For that reason there is an emphasis in this section on the dual cones of cones. Theorem 3.4.25 ahead provides necessary and sufficient conditions for a point $u \in \mathbb{R}^n$ to be in the dual of a cone $K \subset \mathbb{R}^n$. Corollary 3.4.27 applies the conditions of Theorem 3.4.25 to K^* itself to obtain necessary and sufficient conditions for membership in K^{**} . The next section will show that, under certain conditions, K^{**} is just $\text{Conv}(K)$. Thus, Corollary 3.4.27 will provide necessary and sufficient conditions for a point to be in the convexification of a cone; those conditions will be used extensively in Chapter 4.

Several properties of dual cones follow immediately from Definition 3.4.22; these are recorded in the next Proposition for future use.

Proposition 3.4.24. Let $A \subset \mathbb{R}^n$.

- (i) $A \subset A^{**}$
- (ii) If $A \subset B \subset \mathbb{R}^n$, then $B^* \subset A^*$.
- (iii) $A^* = (\text{Conv}(A))^* = (\text{Cone}(A))^* = (\text{Conv}(\text{Cone}(A)))^*$.
- (iv) Let $K_1 = \text{Conv}(\text{Cone}(A))$ and $K_2 = \text{Cone}(A)$. Let B_{K_1} be a base for K_1 , and B_{K_2} a base for K_2 . Then, $A^* = B_{K_1}^* = B_{K_2}^*$.
- (v) If A is a linear subspace of \mathbb{R}^n , then $A^* = A^\perp$.

Proof. The containments in (i) and (ii) follow immediately from Definition 3.4.22.

Part (ii) of the proposition implies that

$$(\text{Conv}(\text{Cone}(A)))^* \subset (\text{Cone}(A))^* \subset A^*$$

and that

$$(\text{Conv}(\text{Cone}(A)))^* \subset (\text{Conv}(A))^* \subset A^*$$

On the other hand, the inclusion $A^* \subset (\text{Conv}(\text{Cone}(A)))^*$ follows from the representation of elements in $\text{Conv}(\text{Cone}(A))$ as linear combinations $\lambda \sum_{i=1}^m a_i a_i$ with $\lambda, a_i \geq 0$ and $a_i \in A$. This proves part (iii).

By part (iii) (applied to B_{K_1}), $B_{K_1}^* = (\text{Conv}(\text{Cone}(B_{K_1})))^*$. But, $\text{Cone}(B_{K_1}) = K_1$. So, $(B_{K_1})^* = (\text{Conv}(K_1))^*$. But, again by part (iii), $(\text{Conv}(K_1))^* = K_1^*$. So, $(B_{K_1})^* = K_1^*$. Appealing to part (iii) a final time yields $A^* = (K_1)^*$. This proves the first equality in (iv). The second follows from a similar argument.

Finally, $A^\perp \subset A^*$ by definition. On the other hand, if A is a linear subspace, and if $k \in A^*$, then $\langle k, a \rangle \geq 0$ and $\langle k, -a \rangle \geq 0$ for all nonzero $a \in A$. Thus, $\langle k, a \rangle = 0$ and $A^* \subset A^\perp$.

□

The following theorem is the main result of this section.

Theorem 3.4.25. *Let $K \subset \mathbb{R}^n$ be a closed cone satisfying $K = \text{Cl}_{\mathbb{R}^n}(\text{Int}_{\mathbb{R}^n}(K))$. Then,*

- (i) *$u \in \mathbb{R}^n$ is in K^* if and only if $\langle u, k \rangle > 0$ for all $k \in \text{Int}_{\mathbb{R}^n}(K)$.*
- (ii) *$u \in \mathbb{R}^n$ is in $\text{Int}_{\mathbb{R}^n}(K^*)$ if and only if $\langle u, k \rangle > 0$ for all nonzero $k \in K$.*
- (iii) *$u \in \mathbb{R}^n$ is in $\partial_{\mathbb{R}^n}(K^*)$ if and only if $u \in K^*$ and there exists a nonzero $k_u \in K$ for which $\langle u, k_u \rangle = 0$.*

Proof. To prove (i), suppose that $u \in \mathbb{R}^n$ satisfies $\langle u, k \rangle = 0$ for some $k \in \text{Int}_{\mathbb{R}^n}(K)$. Then, $\langle u, k - tu \rangle < 0$ for all $t > 0$. But, since $k \in \text{Int}_{\mathbb{R}^n}(K)$, $k - tu$ is in K when $t > 0$ is sufficiently small. Thus, $u \notin K^*$.

Conversely, $\langle u, k \rangle > 0$ for all $k \in \text{Int}_{\mathbb{R}^n}(K)$ if and only if $\text{Int}_{\mathbb{R}^n}(K) \subset \overset{\circ}{H}_u^+$. Therefore, $\text{Cl}_{\mathbb{R}^n}(\text{Int}_{\mathbb{R}^n}(K)) \subset \text{Cl}_{\mathbb{R}^n}(\overset{\circ}{H}_u^+) = H_u^+$. But $\text{Cl}_{\mathbb{R}^n}(\text{Int}_{\mathbb{R}^n}(K)) = K$ by hypothesis, so in fact $K \subset H_u^+$. It follows that $u \in K^*$. This proves (i).

To prove (ii), let $u \in \text{Int}_{\mathbb{R}^n}(K^*)$. By part (i) of the theorem, $\langle u, k \rangle > 0$ for all $k \in K^{**}$. But, by part (i) of Proposition 3.4.24, $K \subset K^{**}$. Thus, $\langle u, k \rangle > 0$ for all $k \in K$.

For the opposite implication, suppose that $u \in \mathbb{R}^n$ satisfies $\langle u, k \rangle > 0$ for all nonzero $k \in K$. Then,

$$\langle u, k \rangle > 0 \quad \text{for all } k \in B_K^{\text{sph}} \quad (3.4.3)$$

B_K^{sph} is compact, since K is closed (by Proposition 3.4.13). The compactness of B_K^{sph} , together with (3.4.3), implies the existence of a neighbourhood $U \subset \mathbb{R}^n$ of u , each of whose members $u' \in U$ satisfy $\langle u', k \rangle > 0$ for all $k \in B_K^{\text{sph}}$. It follows that $u \in \text{Int}_{\mathbb{R}^n}((B_K^{\text{sph}})^*)$. But, $(B_K^{\text{sph}})^* = K^*$. Thus, $u \in \text{Int}_{\mathbb{R}^n}(K^*)$. This completes the proof of (ii).

Part (iii) of the theorem follows from parts (i) and (ii). \square

The following examples show that the assumption $K = \text{Cl}_{\mathbb{R}^n}(\text{Int}_{\mathbb{R}^n}(K))$ in Theorem 3.4.25 cannot be relaxed.

Example 3.4.26. Let $l \subset \mathbb{R}^n$ denote the half-line $\{-\lambda e_n : \lambda \in \mathbb{R}^+\}$. Now let $K \subset \mathbb{R}^n$ be the cone $K = \mathring{H}_{e_n}^+ \cup l$. If $u = e_n$, then $\langle u, k \rangle > 0$ for all $k \in \text{Int}_{\mathbb{R}^n}(K)$. However, $u \notin K^*$. Thus, Part (i) of Theorem 3.4.25 does not necessarily apply to cones that do not satisfy $K = \text{Cl}_{\mathbb{R}^n}(\text{Int}_{\mathbb{R}^n}(K))$.

The same is true of Part (ii) of the theorem. To see this, let $K \subset \mathbb{R}^n$ be the cone $K = \mathring{H}_{e_n}^+ \cup \{0\}$. If $u = e_n$, then $\langle u, k \rangle > 0$ for all nonzero $k \in K$. However, u is not in $\text{Int}_{\mathbb{R}^n}(K^*)$, simply because $K^* = \text{span}\{e_n\}$ has no interior.

The following corollary of Theorem 3.4.25 will play a more significant role in the following sections than Theorem 3.4.25 itself. The corollary follows from the application of Theorem 3.4.25 to K^* when K^* has a nonempty interior.

Corollary 3.4.27. Let $K \subset \mathbb{R}^n$ be a cone and suppose that the interior of K^* is nonempty. Then,

- (i) $u \in \mathbb{R}^n$ is in K^{**} if and only if $\langle u, k \rangle > 0$ for all $k \in \text{Int}_{\mathbb{R}^n}(K^*)$.

(ii) $u \in \mathbb{R}^n$ is in $\text{Int}_{\mathbb{R}^n}(K^{**})$ if and only if $\langle u, k \rangle > 0$ for all nonzero $k \in K^*$.

(iii) $u \in \mathbb{R}^n$ is in $\partial_{\mathbb{R}^n}(K^{**})$ if and only if $u \in K^*$ and there exists a nonzero $k_u \in K^*$ for which $\langle u, k_u \rangle = 0$.

Proof. If $\text{Int}_{\mathbb{R}^n}(K^*) \neq \emptyset$, then by part (iii) of Lemma 3.3.5, $K^* = \text{Cl}_{\mathbb{R}^n}(\text{Int}_{\mathbb{R}^n}(K^*))$. The corollary now follows from Theorem 3.4.25. \square

The next section will establish conditions on the cone K under which $K^{**} = \text{Conv}(K)$. Under those conditions Corollary 3.4.27 will provide necessary and sufficient conditions for membership in $\text{Conv}(K)$.

3.4.3 THE DOUBLE DUAL

The previous section alluded to conditions under which the double dual K^{**} of a cone $K \subset \mathbb{R}^n$ is equal to $\text{Conv}(K)$; this section will establish those conditions (see Theorem 3.4.29 below). When $K^{**} = \text{Conv}(K)$ (and when $\text{Int}_{\mathbb{R}^n}(K^*) \neq \emptyset$), Corollary 3.4.27 can be used to obtain necessary and sufficient conditions for membership in $\text{Conv}(K)$; these are given in Corollary 3.4.30 below.

Let $A \subset \mathbb{R}^n$. A^{**} is a closed convex cone containing A (see Remark 3.4.23). The collection of closed convex cones containing A is closed under intersections. There is therefore a smallest such cone; it turns out that this is A^{**} .

Theorem 3.4.28. *Let $A \subset \mathbb{R}^n$. A^{**} is the smallest closed convex cone containing A .*

The proof of Theorem 3.4.28 is deferred to the end of this section.

If $K \subset \mathbb{R}^n$ is a convex cone, then $\text{Cl}_{\mathbb{R}^n}(K)$ is a closed, convex cone containing K . Thus, by Theorem 3.4.28, $K^{**} \subset \text{Cl}_{\mathbb{R}^n}(K)$. On the other hand, $\text{Cl}_{\mathbb{R}^n}(K)$ is the smallest closed set containing K , so $\text{Cl}_{\mathbb{R}^n}(K) \subset K^{**}$. Thus, when K is a convex cone, K^{**} is just the closure of K .

If K is any cone, convex or not, $K^* = (\text{Conv}(K))^*$; $\text{Conv}(K)$, moreover, is a convex cone. Thus, by the previous paragraph, K^{**} is the closure of $\text{Conv}(K)$: $K^{**} = \text{Cl}_{\mathbb{R}^n}(\text{Conv}(K))$. It follows that $\text{Conv}(K) \subset K^{**}$. However, $\text{Conv}(K)$ and

K^{**} need not be equal in general, simply because $\text{Conv}(K)$ need not be closed, even if K is closed (see [27] for an example).

If $\text{Conv}(K)$ is closed then $\text{Conv}(K) = K^{**}$. This raises a question: when is the convexification of a cone closed? The results of Section 3.4.1 can avail themselves here. By that section's Proposition 3.4.13, $\text{Conv}(K)$ is closed if and only if it has a compact spherical base.

Fortunately, there are easily stated conditions on a cone K under which the spherical base of $\text{Conv}(K)$ is compact. To find those conditions, note that if B_K is a base for K , then $\text{Conv}(B_K)$ generates $\text{Conv}(K)$. If B_K happens to satisfy $0 \notin \text{Conv}(B_K)$, then, by Proposition 3.4.12, the spherical base for $\text{Conv}(K)$ is equal to $r_n(\text{Conv}(B_K))$ (recall that $r_n : \mathbb{R}^n \rightarrow \mathbb{S}^{n-1}$ is the radial retraction map $x \mapsto \frac{x}{\|x\|}$). If B_K happens also to be compact, then so too are $\text{Conv}(B_K)$ and $r_n(\text{Conv}(B_K))$. Weaving this thread together yields the main results of this section.

Theorem 3.4.29. *Let $K \subset \mathbb{R}^n$ be a cone. $\text{Conv}(K)$ is closed if K has a compact base B_K satisfying $0 \notin \text{Conv}(B_K)$. In this case, $\text{Conv}(K) = K^{**}$.*

Corollary 3.4.30. *Let $K \subset \mathbb{R}^n$ be a cone satisfying the conditions of the theorem. Suppose also that $\text{Int}_{\mathbb{R}^n}(K^*) \neq \emptyset$. Then,*

- $k \in \mathbb{R}^n$ is in $\text{Conv}(K)$ if and only if $\langle k, u \rangle > 0$ for all $u \in \text{Int}_{\mathbb{R}^n}(K^*)$.
- $k \in \mathbb{R}^n$ is in $\text{Int}_{\mathbb{R}^n}(\text{Conv}(K))$ if and only if $\langle k, u \rangle > 0$ for all nonzero $u \in K^*$.
- $k \in \mathbb{R}^n$ is in $\partial_{\mathbb{R}^n}(\text{Conv}(K))$ if and only if $k \in \text{Conv}(K)$ and there exists a nonzero $u_k \in K^*$ for which $\langle k, u_k \rangle = 0$.

Proof. This follows from Theorem 3.4.29, and from Corollary 3.4.27. □

The next chapter will show that image of many quadratic maps satisfy the conditions of Corollary 3.4.30. The corollary will be used in the next chapter to describe the convexifications of the images of quadratic maps.

A PROOF OF THEOREM 3.4.28

The proof of Theorem 3.4.28 requires the following separation theorem, which appears as Theorem III.1.3 in [2]. Given a closed convex set and a point not contained in that set, the theorem asserts the existence of an affine hyperplane that separates the point from the set.

Theorem 3.4.31. *If $A \subset \mathbb{R}^n$ is closed and convex and $x \in A^c$, then there exists a $u \in \mathbb{R}^n$ and an $\beta \in \mathbb{R}$ such that $\langle u, x \rangle < \beta$ and $\langle u, a \rangle > \beta$ for all $a \in A$.*

Corollary 3.4.32. *If $K \subset \mathbb{R}^n$ is a closed, convex cone and $x \in K^c$, then there exists a $u \in K^*$ such that $\langle u, x \rangle < 0$.*

Proof. Theorem 3.4.31 applies to K and x . Let $u \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ be the elements provided by the theorem. If $u \notin K^*$, then there would exist a $k \in K$ such that $\langle u, k \rangle < 0$. Then, for some $\lambda \in \mathbb{R}^+$, $\langle u, \lambda k \rangle < \beta$. This would be a contradiction: since $\lambda k \in K$, $\langle u, \lambda k \rangle > \beta$ by the definition of u and β . \square

The proof of Theorem 3.4.28 can now proceed.

Proof. (Proof of Theorem 3.4.28) A^{**} is a closed convex cone containing A . Suppose now that $K \subset \mathbb{R}^n$ is a closed convex cone containing A . If $x \in \mathbb{R}^n$ is any point of K^c , then Theorem 3.4.31 provides a $u \in \mathbb{R}^n$ such that $\langle u, x \rangle < 0$. Since $A \subset K$, u is also in A^* (see part (ii) of Lemma 3.4.24). The fact that $\langle u, x \rangle < 0$ then implies that $x \notin A^{**}$. Thus, A^{**} must be contained in K . This proves the lemma. \square

Remark 3.4.33. A version of Theorem 3.4.31 is still true even if the convex set $A \subset \mathbb{R}^n$ is not closed. Theorem II.2.9 in [2] says that for any point $x \in A^c$, there exists a $u \in \mathbb{R}^n$ and an $\beta \in \mathbb{R}$ satisfying $\langle u, x \rangle = \beta$ and $\langle u, a \rangle > \beta$ for all $a \in A$. When $A = K$ is a convex cone, β can be set to 0, as it was in Corollary 3.4.32. If $K \neq \mathbb{R}^n$ and $x \in K^c$, then there exists a $u \in \mathbb{R}^n$ satisfying $\langle u, x \rangle = 0$ and $\langle u, k \rangle > 0$ for all $k \in K$. Thus, $K^* \neq \{0\}$.

This is a convenient place to prove the following proposition, which is used only in Chapter 7.

Proposition 3.4.34. *Let $K \subset \mathbb{R}^n$ be a cone, and let $P \subset \mathbb{R}^n$ be a linear subspace. If $K^* = P$, then $\text{Conv}(K) = P^\perp$.*

Proof. Given that $K^* = P$, part (v) of Proposition 3.4.24 implies that $K^{**} = P^\perp$. But, as was observed after Theorem 3.4.28, $K^{**} = \text{Cl}_{\mathbb{R}^n}(\text{Conv}(K))$. Thus, $\text{Conv}(K)$ is dense in P^\perp . But then, by Part (vi) of Lemma 3.3.5, $\text{Conv}(K)$ must actually be equal to P^\perp . \square

3.4.4 DEFINITE CONES

The next chapter will show that the image of a definite quadratic map $Q: \mathbb{R}^n \rightarrow \mathbb{R}^m$ has two important properties: it is closed and, apart from $0 \in \mathbb{R}^m$, it is contained in some open half-space $H_u^{++} \subset \mathbb{R}^m$ (equivalently, by Theorem 3.4.25, $u \in \text{Int}_{\mathbb{R}^n}(K^*)$). This section will show that *any* cone K having these two properties satisfies $K^{**} = \text{Conv}(K)$ and is therefore subject to the membership conditions of Corollary 3.4.30. The ubiquity of cones having these two properties in the remainder of this paper warrants a definition.

Definition 3.4.35. *A cone $K \subset \mathbb{R}^n$ is **definite** if it is closed and if $\text{Int}_{\mathbb{R}^n}(K^*) \neq \emptyset$.*

Note that K is definite if and only if $\text{Conv}(K)$ is definite, since $K^* = (\text{Conv}(K))^*$, and since $\text{Conv}(K)$ is closed if K is closed.

Let $K \subset \mathbb{R}^n$ be a definite cone. To see that K satisfies $\text{Conv}(K) = K^{**}$, let $u \in \text{Int}_{\mathbb{R}^n}(K^*)$ (u exists because K is definite). Theorem 3.4.25 implies that $\langle u, k \rangle > 0$ for every nonzero $k \in K$. Let $\alpha > 0$ and define $B_K = H_u^\alpha \cap K$. B_K is flat base for K .

B_K satisfies the two conditions of Theorem 3.4.29: B_K is compact, since K is closed, and $0 \notin \text{Conv}(B_K)$. To prove the latter claim, note that B_K is contained in \mathring{H}_u^+ , which is convex. Thus, $\text{Conv}(B_K)$ is also contained in \mathring{H}_u^+ . However, $0 \notin \mathring{H}_u^+$, and therefore $0 \notin \text{Conv}(B_K)$.

Because K is definite, $\text{Int}_{\mathbb{R}^n}(K^*) \neq \emptyset$ by definition. K thus satisfies the full hypotheses of Corollary 3.4.30. For convenience, Corollary 3.4.30 is restated here in terms of definite cones.

Theorem 3.4.36. *Let $K \subset \mathbb{R}^n$ be a definite cone. Then, $\text{Conv}(K) = K^{**}$. Moreover,*

- (i) *$k \in \mathbb{R}^n$ is in $\text{Conv}(K)$ if and only if $\langle k, u \rangle \geq 0$ for all $u \in \text{Int}_{\mathbb{R}^n}(K^*)$.*
- (ii) *$k \in \mathbb{R}^n$ is in $\text{Int}_{\mathbb{R}^n}(\text{Conv}(K))$ if and only if $\langle k, u \rangle > 0$ for all nonzero $u \in K^*$.*
- (iii) *$k \in \mathbb{R}^n$ is in $\partial_{\mathbb{R}^n}(\text{Conv}(K))$ if and only if $k \in \text{Conv}(K)$ and there exists a nonzero $u_k \in K^*$ for which $\langle k, u_k \rangle = 0$.*

Theorem 3.4.36 will be used in the next chapter to describe the convexification of the image of a definite quadratic map.

3.4.5 FACES OF CONES

Definition 3.4.37. *Let $K \subset \mathbb{R}^n$ be a cone. A nonzero subset $F \subset K$ is a **face** of K if $F = K \cap u^\perp$ for some nonzero $u \in K^*$*

If K is a convex cone, every one of its faces is convex. Like flat bases, faces therefore provide a test for the convexity of a cone. This is recorded in the following proposition. The second part of the proposition says, roughly, that the convexification of a face of a cone is determined by that face alone, independently of the rest of the cone.

Proposition 3.4.38. *Let $K \subset \mathbb{R}^n$ be a cone.*

- (i) *If K has a nonconvex face, K is nonconvex.*
- (ii) *Suppose that $u^\perp \cap K$ is a face of K for some nonzero $u \in K^*$. Then, $\text{Conv}(K) \cap u^\perp = \text{Conv}(K \cap u^\perp)$.*

Proof. Part (i) of the Proposition is clear. For part (ii), note that the inclusion $\text{Conv}(K \cap u^\perp) \subset \text{Conv}(K) \cap u^\perp$ is true by the definition of convexification, since $\text{Conv}(K) \cap u^\perp$ is a convex set containing $K \cap u^\perp$. For the opposite containment, note that $\langle k, u \rangle \geq 0$ for any $k \in K - u^\perp$. It follows that any convex combination of elements of K that lies in $\text{Conv}(K) \cap u^\perp$ must be a convex combination of elements of $K \cap u^\perp$. □

If $u^\perp \cap K$ is a face of K , then u necessarily lies in the boundary $\partial_{\mathbb{R}^n} K^*$ of K^* . Moreover, any point in the face $u^\perp \cap K$ lies in the boundary of K . In general, there may exist points in the boundary $\partial_{\mathbb{R}^n} K$ that are not contained in any face of K . However, when K is a *definite cone*, the faces of $\text{Conv}(K)$ provide a complete description of the boundary $\partial_{\mathbb{R}^n} K$. This description is implicit in part (iii) of Theorem 3.4.36; it will be useful to have it recorded in the terminology of this section.

Proposition 3.4.39. *Let $K \subset \mathbb{R}^n$ be a definite cone. Then, $k \in \partial_{\mathbb{R}^n}(\text{Conv}(K))$ if and only if k lies in some face of K , if and only if $k^\perp \cap K^*$ is a face of K^* . Thus, $\partial_{\mathbb{R}^n}(\text{Conv}(K))$ is the union of all of the faces of $\text{Conv}(K)$,*

$$\begin{aligned}\partial_{\mathbb{R}^n}(\text{Conv}(K)) &= \bigcup_{u \in \partial_{\mathbb{R}^n} K^*, u \neq 0} \text{Conv}(K) \cap u^\perp \\ &= \bigcup_{u \in \partial_{\mathbb{R}^n} K^*, u \neq 0} \text{Conv}(K \cap u^\perp)\end{aligned}$$

Remark 3.4.40. Let $K \subset \mathbb{R}^n$ be a definite, convex cone with $\text{Int}_{\mathbb{R}^n}(K) \neq \emptyset$, and let $P \subset \mathbb{R}^n$ be a linear subspace with $P \cap K \neq \{0\}$, but $P \cap \text{Int}_{\mathbb{R}^n}(K) = \emptyset$. If P is a hyperplane, then necessarily $P = u^\perp$ for some nonzero $u \in \partial_{\mathbb{R}^n} K^*$. If P is not a hyperplane, that is, if $\dim P < n - 1$, then P is nevertheless a subspace of u^\perp for some nonzero $u \in \partial_{\mathbb{R}^n} K^*$. To see this, let V be the quotient space \mathbb{R}^n/P , and let $\pi : \mathbb{R}^n/P \rightarrow V$ be the quotient map. Then, $K' = \pi(K)$ is a convex cone in V , and $\text{Int}_V(K') \neq \emptyset$ (since π is an open map). Since $0 \notin \text{Int}_V(K')$, Remark 3.4.33 implies that there exists a *linear* hyperplane $H' \subset V$ with $H' \cap \text{Int}_V(K') = \emptyset$. The preimage $H = \pi^{-1}(H')$ is then a linear hyperplane in \mathbb{R}^n satisfying $P \subset H$, $H \cap \text{Int}_{\mathbb{R}^n}(K) = \emptyset$ and $H \cap K \neq \{0\}$. Thus, $H = u^\perp$ for some nonzero $u \in \partial_{\mathbb{R}^n} K^*$, and it contains P . ■

3.4.6 S_n^+

This section records a few basic facts about S_n^+ , the cone of positive semidefinite matrices in Sym_n .

Proposition 3.4.41. S_n^+ has the following properties.

- (i) S_n^+ is self-dual: $S_n^+ = (S_n^+)^*$.
- (ii) $M \in \partial S_n^+$ if and only if $S_n^+ \cap M^\perp$ is a face of S_n^+ .
- (iii) If $M \in \partial S_n^+$ has rank $n - 1$, $S_n^+ \cap M^\perp = \text{Cone}(xx^T)$ where x is any nonzero element of $\ker M$.

Proof. The proof of (i) is straightforward; it can be found in [2]. Part (ii) of the proposition follows from part (i) and Proposition 3.4.39 in the previous section.

To prove part (iii), suppose that $M' \in S_n^+ \cap M^\perp$, and write M' as $M' = \sum_{i=1}^k x_i x_i^T$ for some $x_i \in \mathbb{R}^n$. Then, $0 = \langle M, M' \rangle_{\text{Sym}_n} = \sum_{i=1}^k \langle x_i, M x_i \rangle_{\mathbb{R}^n}$. The positive semidefiniteness of M implies that $\langle x_i, M x_i \rangle_{\mathbb{R}^n} = 0$ for all i ; each x_i lies in the 1-dimensional subspace $\ker M$. This completes the proof. \square

3.4.7 PULLBACK CONES

Let V and W be vector spaces and let $\psi : V \rightarrow W$ be a linear map.

Definition 3.4.42. Let $K \subset W$ be a cone. $\psi^{-1}(K)$ is a cone in V and is called the *pullback cone of K by ψ* .

Proposition 3.4.43. If $K \subset W$ is a convex cone, so is $\psi^{-1}(K)$.

Section 4.1 will show that every quadratic map $Q : V \rightarrow W$ is uniquely associated with a linear map $\psi_Q : W \rightarrow \text{Sym}(V)$. The preimage of $S^+(V)$ under ψ_Q plays a significant role in the study of the convexity of Q .

The following proposition will be used only in Section 7.3.

Proposition 3.4.44. Let $K \subset W$ be a cone. Then, $\psi^{-1}(K^*) = (\psi^*(K))^*$, where ψ^* denotes the adjoint of ψ . In particular, if $K = K^*$ then $\psi^{-1}(K) = (\psi^*(K))^*$.

Proof. Let $v \in V$. Then,

$$\begin{aligned}
v &\in (\psi^*(K))^* \\
&\Leftrightarrow \langle v, \psi^*(k) \rangle_V \geq 0 \quad \forall k \in K \\
&\Leftrightarrow \langle \psi(v), k \rangle_V \geq 0 \quad \forall k \in K \\
&\Leftrightarrow \psi(v) \in K^* \\
&\Leftrightarrow v \in \psi^{-1}(K^*)
\end{aligned}$$

□

Corollary 3.4.45. Let $\psi : \mathbb{R}^m \rightarrow \text{Sym}_n$ be a linear map. Then,
 $\psi^{-1}(S_n^+) = (\psi^*(S_n^+))^*$.

3.5 SUBSPACES OF SYM_n

Taking the preimage of a point under a quadratic map involves solving a homogeneous system of multivariate quadratic equations (see Lemma 4.2.9). This section establishes some basic facts about such systems. Throughout this section V will denote an inner product space.

Definition 3.5.1. Let $k \in \mathbb{Z}^+$. A k -tuple (M_1, \dots, M_k) with $M_i \in \text{Sym}(V)$ is *nice* if the following system of equations has a nonzero solution:

$$\begin{aligned}
\langle x, M_1 x \rangle &= 0 \\
&\vdots \\
\langle x, M_k x \rangle &= 0
\end{aligned} \tag{3.5.1}$$

Remark 3.5.2. Recall that for $x \in V$, $x \otimes x \in \text{Sym}(V)$ denotes the map $w \mapsto \langle x, w \rangle x$. The identity $\langle x \otimes x, M \rangle_{\text{Sym}(V)} = \langle x, Mx \rangle_V$ implies that a k -tuple (M_1, \dots, M_k) is nice if and only if $\cap_{i=1}^k M_i^\perp$ contains $x \otimes x$ for some nonzero $x \in V$. ■

The system (3.5.1) is linear the M_i 's. If the k -tuple (M_1, \dots, M_k) is nice, so too is any k' -tuple $(M'_1, \dots, M'_{k'})$ whose components M'_i are linear combinations of the M_i . Niceness, therefore, is a property that can be meaningfully attributed to subspaces of $\text{Sym}(V)$, not just tuples in $\text{Sym}(V) \times \dots \times \text{Sym}(V)$.

Definition 3.5.3. Let $P \subset \text{Sym}(V)$ be a linear subspace. P is **nice** if there exists a nonzero $x \in V$ which is such that $\langle x, Mx \rangle = 0$ for all $M \in P$.

Proposition 3.5.4. Let $P \subset \text{Sym}(V)$ be a linear subspace. P is nice if and only if every k -tuple (M_1, \dots, M_k) with $M_i \in P$ is nice, for all $k \in \mathbb{Z}^+$. In particular, P is nice if and only if $(M_1, \dots, M_{\dim P})$ is nice for some basis $\{M_1, \dots, M_{\dim P}\}$ of P .

Remark 3.5.5. As was the case with tuples, a subspace $P \subset \text{Sym}(V)$ is nice if and only if P^\perp contains $x \otimes x$ for some nonzero $x \in V$. Equivalently, P is nice if and only if $P \subset (x \times x)^\perp$ for some nonzero $x \in V$. ■

It will be useful to have slightly more succinct notation for niceness.

Definition 3.5.6. Let $M \in \text{Sym}(V)$. The **zero set** of M is the subset \mathcal{Z}_M . If $C \subset \text{Sym}(V)$, the **zero set** of C is the set

$$\mathcal{Z}_C = \{x \in V \mid \langle x, Mx \rangle = 0 \ \forall M \in C\} = \bigcap_{M \in C} \mathcal{Z}_M$$

In the terms of Definition 3.5.6, a subspace $P \subset \text{Sym}(V)$ is nice if and only if $\mathcal{Z}_P \neq \{0\}$.

In addition to being nice or not nice, subspaces of $\text{Sym}(V)$ can be classified according to their intersection with $S^+(V)$.

Definition 3.5.7. Let $P \subset \text{Sym}(V)$ be a linear subspace.

- P is **indefinite** if $P \cap S^+(V) = 0$,
- P is **definite** if $P \cap \mathring{S}^+(V) \neq \emptyset$,
- P is **semidefinite** if $P \cap S^+(V) \neq 0$ and $P^\perp \cap S^+(V) = 0$.

So, a subspace is indefinite if each of its nonzero elements is indefinite, definite if it contains a positive definite element, and semidefinite if it contains a positive semidefinite element but no positive definite elements.

Remark 3.5.8. Let $P \subset \text{Sym}_n$ be a subspace that contains a nonzero positive semidefinite matrix, but does not contain any positive definite matrices. By Remark 3.4.40, $P \subset M^\perp$, where $M \in \partial S_n^+$ is a nonzero positive semidefinite matrix. ■

The relationship between niceness and the three properties in Definition 3.5.7 plays a fundamental role in the study of the convexity of quadratic maps. For example, definiteness is a sufficient condition for a subspace P to fail to be nice. This is because $\mathcal{Z}_P \subset \mathcal{Z}_M$ for all $M \in P$, and $\mathcal{Z}_M = \{0\}$ for all $M \in \mathring{S}^+(V)$. When $\dim P = 2$, definiteness is also a *necessary* condition for the failure of P to be nice. This is due to a theorem of Calabi. It is designated here as Calabi's Niceness Theorem to distinguish it from Calabi's Convexity Theorem, Theorem 2.1.5.

Theorem 3.5.9 ((Calabi's Niceness Theorem) Calabi, [9]). *Let $P \subset \text{Sym}(V)$ be a subspace with $\dim P \leq 2$. Then, P is nice if and only if P is not definite.*

When $\dim P > 2$, there is no steadfast relationship between the definiteness of P and its niceness; an indefinite subspace of $\text{Sym}(V)$ may be nice, or it may not, and the same is true of semidefinite subspaces (see Examples 3.5.12 and 3.5.13 below). The failure of Theorem 3.5.9 to hold when $\dim P > 2$ is the whole reason that Calabi's Convexity Theorems, Theorems 2.1.5 and 2.2.2, fail to hold when $m > 3$. The existence of indefinite subspaces of $\text{Sym}(V)$ that are *not* nice ensures the existence of nonconvex quadratic maps (see Example 4.4.14).

As mentioned above (and as demonstrated ahead in Example 3.5.13), if $P \subset \text{Sym}_n$ is a semidefinite subspace with $\dim P > 2$, P is not necessarily nice. However, if P contains an $M \in \partial S_n^+$ with $\text{rank } M = n - 1$, then P is nice.

Proposition 3.5.10. *Let $P \subset \text{Sym}_n$ be a semidefinite subspace and suppose that there is an $M_0 \in P \cap \partial S_n^+$ with $\text{rank } M_0 = n - 1$. Then, P is nice and $\mathcal{Z}_P = \ker M_0$.*

Proof. Because P is semidefinite, Remark 3.5.8 implies that, $P \subset M^\perp$ for some nonzero $M \in \partial S_n^+$. By part (iii) of Proposition 3.4.41, the orthogonality of M and M_o implies that $M = xx^T$ for some nonzero $x \in \ker M_o$. Thus, $P \subset (xx^T)^\perp$, which makes P nice. \square

Proposition 3.5.10 is vital to the Roundness Theorem; it explains why a round quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies the containment $\partial_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q)) \subset \text{Im } Q$.

Remark 3.5.11. Something can still be said for those semidefinite subspaces $P \subset \text{Sym}_n$ that do not contain a rank $n - 1$ $M \in \partial S_n^+$, namely that \mathcal{Z}_P is contained in the subspace $\cap_{M \in \partial S_n^+ \cap P} \ker M$. ■

This section concludes with two examples. The first is an indefinite subspace of Sym_4 that is not nice, and the second is a semidefinite subspace of Sym_4 that is also not nice.

Example 3.5.12. Let $P = \text{span} \{M_1, M_2, M_3\}$ where

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Each M_i is traceless. So too, therefore, is every element of P . Thus, P is indefinite.

However, P is not nice. The equations $\langle x, M_1 x \rangle = 0$ and $\langle x, M_2 x \rangle = 0$ force the first two components of $x \in \mathbb{R}^4$ to be zero. But then the equation $\langle x, M_3 x \rangle = 0$ forces the last two components of x to also be zero. Thus, there are no nonzero solutions to $\langle x, M_1 x \rangle = \langle x, M_2 x \rangle = \langle x, M_3 x \rangle = 0$.

Example 3.5.13. Let $P = \text{span} \{M_1, M_2, M_3\}$ where

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The upper left 2×2 block of every matrix in P is traceless; P cannot be definite. But, P contains M_3 , which is semidefinite. P , therefore, is semidefinite.

However, by the same argument that was given in Example 3.5.12, P is also not nice.

3.6 THE COMPACT-OPEN TOPOLOGY

Let $\mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of quadratic maps from \mathbb{R}^n to \mathbb{R}^m . By associating a quadratic map $Q \in \mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ with the tuple (M_1, \dots, M_m) that defines it, $\mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ can be identified with the m -fold product $\text{Sym}_n \times \dots \times \text{Sym}_n$. This identification endows $\mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ with the topology of $\text{Sym}_n \times \dots \times \text{Sym}_n$. With that topology in place, Definition 2.2.1, the definition of stable convexity, can be stated as this: a quadratic map $Q \in \mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ is stably convex if and only if it is in the *interior* of the set of *convex* quadratic maps relative to the full set of quadratic maps $\mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$.

Definition 2.2.1, the original definition of stable convexity, has a shortcoming: it relies on the identification of $\mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ with the m -fold product $\text{Sym}_n \times \dots \times \text{Sym}_n$. In the next chapter quadratic maps will be defined as mapping from one inner product space, V , to another, W , without reference to a choice of coordinates on either. This will render Definition 2.2.1 obsolete. However, so long as $\mathcal{Q}(V, W)$, the set of quadratic maps from V to W , is given a topology, the notion of stable convexity can be salvaged. If $\mathcal{Q}(V, W)$ has a topology, a quadratic map $Q \in \mathcal{Q}(V, W)$ can be defined as being stably convex if it is in the interior of the set of *convex* quadratic maps, relative to $\mathcal{Q}(V, W)$.

Fortunately, there is a natural topology on $\mathcal{Q}(V, W)$, namely the compact-open topology. The compact-open topology is a standard topology on the set of maps from one topological space to another.

Definition 3.6.1. Let X and Y be topological spaces. Let Y^X denote the set of functions from X to Y . The **compact-open topology** on Y^X is the topology generated by the subsets $B(K, U) = \{f \in Y^X \mid f(K) \subset U\}$, where $K \subset X$ varies over compact subsets of X and $U \subset Y$ varies over open subsets of Y . This topology is denoted by τ_{co} .

A subset $Q \subset Y^X$ is said to have the **compact-open topology** if it has the subspace topology induced by τ_{co} . Thus, the compact-open topology on Q is the generated by the collection $\{B(K, U) \cap Q \mid K \subset X \text{ compact, } U \subset Y \text{ open}\}$.

The compact-open topology will prove useful in the next chapter; it will simplify arguments about the topology of various subsets of $\mathcal{Q}(V, W)$. For example, the stability property in Proposition 3.6.3 below will be used to show that the set of round maps in $\mathcal{Q}(V, W)$ is open; the proposition follows immediately from the definition of the compact-open topology.

The first proposition in this section establishes that compact-open topologies can be generated by a collection of sets that is potentially smaller than the one described in Definition 3.6.1. In the following propositions, X and Y still denote topological spaces.

Proposition 3.6.2. *Let S be a subbase for the topology on Y and let $Q \subset \mathcal{C}(X, Y)$. Then, the collection $\{B(K, U) \cap Q \mid K \subset X \text{ compact, } U \in S\}$ also generates the compact-open topology on Q .*

Proof. See 5.1 in Chapter XII of [15]. □

The next proposition follows immediately from the definition of the compact-open topology. Informally, it says that the intersection of the image of a continuous map with a closed subset varies continuously as the map varies.

Proposition 3.6.3. *Let X and Y be topological spaces, $K \subset X$ a compact subset, $C \subset Y$ a closed subset, and $U \subset Y$ an open neighbourhood of C . Suppose that Y^X , the space of maps from X to Y has the compact-open topology. Then, in the notation of Definition 3.6.1, the open set $B(K, U \cup C^c) \subset Y^X$ consists of maps f for which $f(K) \cap C \subset U \cap C$.*

Proposition 3.6.3 will be used in the next chapter to show that the set of round quadratic maps from \mathbb{R}^n to \mathbb{R}^m is an open subset of the set of all quadratic maps from \mathbb{R}^n to \mathbb{R}^m .

The final proposition in this section has to do with the topology of $\mathcal{L}(V, W)$, the set of linear maps from one real inner product space V to another, W . There are two ways to define a topology on $\mathcal{L}(V, W)$: one using the fact that $\mathcal{L}(V, W)$ is itself a real vector space, and the other using the compact-open topology. It turns out that these topologies are the same.

Proposition 3.6.4. *Let V and W be finite-dimensional, real inner-product spaces. The vector space topology on $\mathcal{L}(V, W)$ coincides with its compact-open topology.*

The proof of Proposition 3.6.4 does not bear on any other part of this thesis; it can safely be skipped.

Proof. (Proof of Proposition 3.6.4) The vector space topology on $\mathcal{L}(V, W)$ is the same as that induced by the norm $\|T\| = \sup_{x \in S^V} \|Tx\|$ (recall that S^V is the unit sphere in V). Let T be an element of $\mathcal{L}(V, W)$ and let $B_T(\varepsilon) = \{S \in \mathcal{L}(V, W) \mid \|S - T\| < \varepsilon\}$ be the open ε -ball centered at T . Then, in the notation of Definition 3.6.1, $B_T(\varepsilon) = B(S^V, U_\varepsilon)$ where U_ε is the ε -neighbourhood of $T(S^V)$,

$$U_\varepsilon = \bigcup_{x \in S^V} B_{T(x)}(\varepsilon)$$

In the preceding equation, $B_{T(x)}(\varepsilon)$ denotes the ε -ball around $T(x)$ in W . It follows that the vector space topology is contained in the compact-open topology.

For the opposite containment, let $B(K, U)$ be a subbasic element of the compact-open topology (so, $K \subset V$ is compact and $U \subset W$ is open). Let $T \in B(K, U)$. Because $T(K) \subset W$ is compact and contained in U , there exists an ε such that $U_\varepsilon^{T(K)}$, the ε -neighbourhood of $T(K)$, is contained in U (for example, ε could be the Lebesgue number of the open covering U of K). So, $B_T(\varepsilon) \subset B(K, U)$, from which it follows that $B(K, U) = \bigcup_{T \in B(K, U)} B_T(\varepsilon)$. This proves that the compact-open topology is contained in the vector space topology. \square

3.7 PROPER MAPS

Let V and W be finite-dimensional, real vector spaces. A map $f : V \rightarrow W$ is *k-homogeneous* if there exists a $k \in \mathbb{Z}$ for which $f(\lambda x) = \lambda^k f(x)$, for all $x \in V$, and for all $\lambda \in \mathbb{R}$. For example, quadratic maps are 2-homogeneous. This section proves that the image of a homogeneous map $f : V \rightarrow W$ is closed if and only if f satisfies $f^{-1}(0) = \{0\}$. This fact will be used to show that the image of a definite quadratic map is closed. The proof invokes the notion of properness.

Definition 3.7.1. Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is **proper** if $f^{-1}(C)$ is compact for every compact subset $C \subset Y$.

Proposition 3.7.2. Let X and Y be locally compact Hausdorff spaces. If the map $f : X \rightarrow Y$ is proper, then $\text{Im } f$ is a closed subset of Y .

Proof. See [35]. □

Corollary 3.7.3. Let V and W be finite-dimensional real vector spaces. If the map $f : V \rightarrow W$ is proper, then $\text{Im } f$ is a closed subset of W .

Because of Corollary 3.7.3, the next proposition implies the result promised at the outset of this section.

Proposition 3.7.4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous, *k-homogeneous* map. Then, f is proper if and only if $f^{-1}(0) = \{0\}$.

Proof. If $f^{-1}(0)$ contains a nonzero $x \in \mathbb{R}^n$, then, by the *k-homogeneity* of f , $f^{-1}(0)$ contains the subspace generated by x , $\langle x \rangle = \{\lambda x \mid \lambda \in \mathbb{R}\}$ and is therefore not compact. In this case, f cannot be proper: the preimage of the compact subspace $\{0\}$ would contain the noncompact subspace $\langle x \rangle$. So, if f is proper, then $f^{-1}(0) = \{0\}$.

Suppose now that $f^{-1}(0) = \{0\}$. The image of the unit sphere \mathbb{S}^{n-1} under f is then a compact subspace of \mathbb{R}^m that does not contain $0 \in \mathbb{R}^m$. The minimum norm $m = \min\{\|f(x)\|_{\mathbb{R}^m} \mid x \in \mathbb{S}^{n-1}\}$ is therefore strictly greater than zero. That is, $0 < m \leq \|f(x)\|$ for all $x \in \mathbb{S}^{n-1}$. The homogeneity of both f and the norm on

V imply that $0 < m\|x\|^k \leq \|f(x)\|$ for all nonzero $x \in \mathbb{R}^n$. This inequality implies that the preimage under f of a bounded set in \mathbb{R}^m is bounded in \mathbb{R}^n . The continuity of f now implies that the preimage of a closed and bounded set in \mathbb{R}^m is closed and bounded in \mathbb{R}^n . This proves the proposition. \square

3.8 DIFFERENTIAL GEOMETRY

This section presents results from differential geometry that are invoked in the proof of the Roundness Theorem. The material in this section requires some familiarity with elementary aspects of differential geometry, specifically these: the definition of a manifold, of a submanifold, of tangent spaces, of smooth maps and their differentials, of diffeomorphisms and embeddings, of the projective space \mathbb{RP}^{n-1} , of Lie groups and of quotient groups, and finally of transitive group actions and their stabilizers. For a reference, see [13] or [43].

NOTATION

Throughout this chapter and the next, the term manifold will refer exclusively to smooth, or C^∞ , manifolds. The term submanifold has a variety of meanings. Here it will refer exclusively to what are usually called embedded submanifolds (see Definition 3.7.1 in [13]).

If M is a manifold and $p \in M$, the tangent space at p will be denoted by $T_p M$. If N is another manifold and $f : M \rightarrow N$ a smooth map, $f_{*,m} : T_m M \rightarrow T_{f(m)} N$ will denote the Jacobian of f at $m \in M$.

If V is a finite-dimensional vector space and $p \in V$, $T_p V$ will be identified with V in the usual manner (see [43] for details).

Finally, $\pi : \mathbb{S}^{n-1} \rightarrow \mathbb{RP}^{n-1}$ will denote the map sending $x \in \mathbb{S}^{n-1}$ to the 1-dimensional subspace of \mathbb{R}^n that is spanned by x . If $x \in \mathbb{R}^n - \{0\}$ is nonzero, then the element of \mathbb{RP}^{n-1} corresponding to $x/\|x\| \in \mathbb{S}^{n-1}$ will be denoted by $[x]$.

3.8.1 CONVEX BODIES

Chapters 4 and 5 will show that a round quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has the following properties,

1. $Q(\mathbb{S}^{n-1})$, the image of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ under Q is convex.
2. The interior of $Q(\mathbb{S}^{n-1})$, relative to some affine hyperplane $H_u^a \subset \mathbb{R}^m$ is nonempty.

Thus, when Q is round, $Q(\mathbb{S}^{n-1})$ is a compact, convex subset of H_u^a whose interior is nonempty. Such subsets have a special name.

Definition 3.8.1. A subset $C \subset \mathbb{R}^n$ is a **convex body** if it is convex, compact, and if $\text{Int}_{\mathbb{R}^n}(C) \neq \emptyset$.

Proposition 3.8.2. Let $C \subset \mathbb{R}^n$ be a convex body. Then, C is homeomorphic to the closed unit disk \mathbb{D}^n and $\partial_{\mathbb{R}^n} C$ is homeomorphic to \mathbb{S}^{n-1} .

Proof. See section 1.16 of [5]. □

The book [5] explains that the homeomorphism from $\partial_{\mathbb{R}^n} C$ to \mathbb{S}^{n-1} alluded to in the proposition is the map $r_n \circ T \circ i$, where $i : \partial_{\mathbb{R}^n} C \hookrightarrow \mathbb{R}^n$ is the inclusion of $\partial_{\mathbb{R}^n} C$, $r_n : \mathbb{R}^n - \{o\} \rightarrow \mathbb{S}^{n-1}$ the radial retraction and $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ any translation of \mathbb{R}^n which is such that $o \in \text{Int}_{\mathbb{R}^n} T(C)$.

The maps r_n and T are smooth. If $\partial_{\mathbb{R}^n} C$ is a smooth submanifold of \mathbb{R}^n , then the inclusion map i is smooth as well. In this event, the homeomorphism $r_n \circ T \circ i$ is in fact a diffeomorphism. This proves the next proposition.

Proposition 3.8.3. If $C \subset \mathbb{R}^n$ is a convex body and $\partial_{\mathbb{R}^n} C$ a submanifold of \mathbb{R}^n , then $\partial_{\mathbb{R}^n} C$ is diffeomorphic to \mathbb{S}^{n-1} .

Proposition 3.8.3 will be used in the Chapter 4 to show that the boundary of the image of a round quadratic map is a cone that has a flat base that is diffeomorphic to \mathbb{S}^{n-1} .

3.8.2 COVERING SPACES

Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map. For the purposes of this section, let B denote the boundary of $\text{Im } Q$ apart from the origin, $B = \partial_{\mathbb{R}^m}(\text{Im } Q) - \{0\}$, and let T denote the preimage of B under Q , $T = Q^{-1}(B) \subset \mathbb{R}^n$. A critical step in the proof of the Roundness Theorem shows that the restriction $Q|_T : T \rightarrow B$ is a covering map. This step calls on a handful of facts from the theory of covering spaces. This section provides a hasty introduction to covering spaces. A thorough treatment can be found in [38].

Definition 3.8.4. Let T and B be topological spaces with B Hausdorff, locally-path connected and path-connected. A map $p : T \rightarrow B$ is a **covering map** (or simply a **covering**) if for every $b \in B$ there exists an open neighbourhood $U \subset B$ of b which is such that $p^{-1}(U)$ is a union of disjoint open subsets of T , each of which is mapped homeomorphically to U by p . For $b \in B$, the preimage $p^{-1}(b)$ is the **fiber** over b .

A covering $p : T \rightarrow B$ for which each fiber $p^{-1}(b)$, with $b \in B$, contains m points ($m \in \mathbb{Z}^+$) will be called an **m -sheeted cover**. The **trivial m -sheeted covering** of B is the map $p : B \times \{0, \dots, m-1\} \rightarrow B$ sending $(b, k) \mapsto b$.

Two coverings $p_1 : T_1 \rightarrow B$ and $p_2 : T_2 \rightarrow B$ are **equivalent** if there is a homeomorphism $h : T_1 \rightarrow T_2$ which is such that $p_2 \circ h = p_1$.

The next proposition follows at once from Definition 3.8.4.

Proposition 3.8.5. Let $p : T \rightarrow B$ be an m -sheeted covering and let $S \subset B$ be a locally path-connected, path-connected topological subspace (that is, a subset of B with the subspace topology). Then, $p|_{p^{-1}(S)} : p^{-1}(S) \rightarrow S$ is an n -sheeted covering.

Definition 3.8.6. Let $p : T \rightarrow B$ be a covering and let $S \subset B$ be a locally-path connected, path-connected subspace. The covering $p|_{p^{-1}(S)} : p^{-1}(S) \rightarrow S$ is **induced over S by p** .

Remark 3.8.7. If $p : T \rightarrow B$ is a covering, and B happens to be a manifold, then there is a unique smooth structure on T which is such that p is a smooth map. If $S \subset B$ happens to be a submanifold of B , then the differential structure on S

induces one on $p^{-1}(S)$ that makes $p^{-1}(S)$ a submanifold of T ; the induced covering $p|_{p^{-1}(S)}$ is smooth. ■

The most important covering map in this section is the quotient map $\pi : \mathbb{S}^{n-1} \rightarrow \mathbb{RP}^{n-1}$. It is a smooth, 2-sheeted covering map.

The main result of this section is about induced coverings, induced by $\pi : \mathbb{S}^{n-1} \rightarrow \mathbb{RP}^{n-1}$, of smoothly embedded spheres \mathbb{S}^k in \mathbb{RP}^{n-1} . The nature of these induced coverings is simplified by a basic fact about coverings of \mathbb{S}^k : when $k \geq 2$, every m -sheeted covering of \mathbb{S}^k is equivalent to the trivial m -sheeted covering $\mathbb{S}^k \times \{0, \dots, m-1\} \rightarrow \mathbb{S}^k$ (see [38], for example, for a detailed explanation). If $S \subset \mathbb{RP}^{n-1}$ is diffeomorphic to a sphere \mathbb{S}^k , with $2 \leq k < n-1$, it follows that $\pi^{-1}(S)$ is the disjoint union $\bar{S} \amalg -\bar{S}$, where $\bar{S} \subset \mathbb{S}^{n-1}$ is diffeomorphic to \mathbb{S}^k , and $-\bar{S}$ the image of \bar{S} under the antipodal map of \mathbb{S}^{n-1} . This actually proves Lemma 3.8.8.

Lemma 3.8.8. *Suppose that $S \subset \mathbb{RP}^{n-1}$ is diffeomorphic to \mathbb{S}^k with $2 \leq k < n-1$. Then, $\pi^{-1}(S) = \bar{S} \amalg -\bar{S}$, where \bar{S} is diffeomorphic to \mathbb{S}^k and $-\bar{S}$ is the image of \bar{S} under the antipodal map of \mathbb{S}^{n-1} .*

3.8.3 THE SMOOTH STRUCTURE OF THE SET OF RANK $n-1$ SEMIDEFINITE MATRICES

Definition 3.8.9. $S_{n,k}^+$ is the set of rank k positive semidefinite matrices in Sym_n .

$S_{n,n-1}^+$ plays a significant role in the study of round quadratic maps; a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by the symmetric matrices M_1, \dots, M_m is round if and only if $(\text{span} \{M_1, \dots, M_m\}) \cap (\partial S_n^+ - \{0\}) \subset S_{n,n-1}^+$.

The kernel of each $M \in S_{n,n-1}^+$ is a 1-dimensional subspace of \mathbb{R}^n . The map $k : S_{n,n-1}^+ \rightarrow \mathbb{RP}^{n-1}$ taking $M \in S_{n,n-1}^+$ to its kernel is of particular importance in the proof of the Roundness Theorem. The primary purpose of this section is to show that k is a smooth map. This, of course, presumes that $S_{n,n-1}^+$ is a manifold. This is discussed first.

Proposition 3.8.10. $S_{n,n-1}^+$ is a codimension 1 submanifold of Sym_n . Under the usual identification of $T_M \text{Sym}_n$ with Sym_n , the tangent space $T_M S_{n,n-1}^+$ to $S_{n,n-1}^+$ at M is $(xx^T)^\perp$, where $x \in \mathbb{R}^n$ is any nonzero element of $\ker M$.

Proof. See [42]. □

Remark 3.8.11. $S_{n,n-1}^+$ admits a smooth transitive action of GL_n . This action is given by $M \mapsto AMA^T$ for $M \in S_{n,n-1}^+$ and $A \in GL_n$. Thus, $S_{n,n-1}^+$ is diffeomorphic to the homogeneous space GL_n/Stab_{M_0} where Stab_{M_0} is the stabilizer of

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in S_{n,n-1}^+$$

under the given action (see [42]). Note that

$$\text{Stab}_{M_0} = \left\{ \begin{pmatrix} \theta & b \\ 0 & a \end{pmatrix} \mid \theta \in SO(n-1), b \in \mathbb{R}^n, a \in \mathbb{R} - \{0\} \right\}$$

It follows that for all $A \in \text{Stab}_{M_0}$, $e_n \in \mathbb{R}^n$ is an eigenvector of A^T . This fact will be called on in a moment.

Note, finally, that if

$$A = \begin{pmatrix} \theta & b \\ 0 & a \end{pmatrix}$$

then,

$$A^{-1} = \begin{pmatrix} \theta^{-1} & -a^{-1}\theta^{-1}b \\ 0 & a^{-1} \end{pmatrix} \quad (3.8.1)$$

■

For the next proposition, recall that if G is a Lie group and $H \subset G$ a closed

subgroup of G , then G/H is a smooth manifold (see [43]).

Proposition 3.8.12. *Let G be a Lie group and $H \subset G$ a closed subgroup of G . Let N be a manifold. A function $f : G/H \rightarrow N$ is smooth if and only if the map $f \circ p : G \rightarrow N$ is smooth where $p : G \rightarrow G/H$ is the quotient map.*

Proof. See [43] □

Lemma 3.8.13. *Let $k : S_{n,n-1}^+ \rightarrow \mathbb{RP}^{n-1}$ denote the map $M \mapsto \ker M$. Then, k is smooth.*

Proof. The function $\hat{k} : GL_n \rightarrow \mathbb{RP}^{n-1}$ given by $A \mapsto [(A^T)^{-1}e_n]$ is smooth. It, moreover, is constant on left cosets of Stab_{M_0} : if $S \in GL_n$ and $A \in \text{Stab}_{M_0}$ with

$$A = \begin{pmatrix} \theta & b \\ 0 & a \end{pmatrix}$$

then $((SA)^T)^{-1}e_n = (S^T)^{-1} \cdot (A^T)^{-1}e_n = a^{-1}(S^T)^{-1}e_n$, where the final equality follows from the formula for A^{-1} in equation (3.8.1). Thus,

$$[((SA)^T)^{-1}e_n] = [(S^T)^{-1}e_n].$$

It follows that \hat{k} descends to a function $\bar{k} : GL_n/\text{Stab}_{M_0} \rightarrow \mathbb{RP}^{n-1}$ on the quotient group GL_n/Stab_{M_0} . By Proposition 3.8.12, \bar{k} is smooth.

By Remark 3.8.11, there is a diffeomorphism from $S_{n,n-1}^+$ to GL_n/Stab_{M_0} sending AM_0A^T to the coset $[A] \in GL_n/\text{Stab}_{M_0}$, where $A \in GL_n$. The kernel of AM_0A^T is just $(A^T)^{-1}e_n$. It follows the composition of \bar{k} with the diffeomorphism between $S_{n,n-1}^+$ and GL_n/Stab_{M_0} takes AM_0A^T to $\ker AM_0A^T$. This is exactly the map k . k , therefore, is smooth. □

There will be a need in the proof of the Roundness Theorem to understand the differential $k_{*,M}$ of k at a point $M \in S_{n,n-1}^+$. This needs a brief foreword on the tangent space $T_M S_{n,n-1}^+$.

Lemma 3.8.14. *Under the identification of $T_M \text{Sym}_n$ with Sym_n , the tangent space of $S_{n,n-1}^+$ at $M \in S_{n,n-1}^+$ is the following subspace,*

$$T_M S_{n,n-1}^+ = \{B^T M + MB : B \in \mathfrak{gl}_n\}$$

(where \mathfrak{gl}_n is the set of real $n \times n$ matrices).

Proof. Because the action of GL_n on $S_{n,n-1}^+$ is transitive, every tangent vector at $M \in S_{n,n-1}^+$ can be represented as the derivative of a curve of the form,

$$t \in \mathbb{R} \mapsto e^{Bt} M e^{B^T t}$$

where $e : \mathfrak{gl}_n \rightarrow GL_n$ denotes the matrix exponential. Taking the derivative of such a curve at $t = 0$ will yield the lemma. \square

Proposition 3.8.15. *The Jacobian of k at a point $M \in S_{n,n-1}^+$ has the following kernel,*

$$\ker k_{*,M} = \{BM + MB^T : B \in \mathfrak{gl}_n, \ker B \supset \ker M\}$$

Proof. Let $B \in \mathfrak{gl}_n$ and consider the curve $c(t) = e^{Bt} M e^{B^T t}$. Then,

$$\begin{aligned} k(c(t)) &= \ker e^{Bt} M e^{B^T t} \\ &= e^{-B^T t} \ker M \end{aligned} \tag{3.8.2}$$

where $e^{-B^T t} \ker M$ is meant to denote the 1-dimensional subspace $\{e^{-B^T t} x : x \in \ker M\}$. Taking the derivative of (3.8.2) at $t = 0$ yields this,

$$\begin{aligned} k_{*,M}(\dot{c}(0)) &= BM + MB^T \\ &= -B^T \ker M \end{aligned}$$

Thus, if $k_{*,M}(\dot{c}(0)) = 0$, then $\ker M \subset \ker B^T$. \square

3.8.4 TRANSVERSALITY

The previous section observed that a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $M_1, \dots, M_m \in \text{Sym}_n$ is round if and only if

$(\text{span}\{M_1, \dots, M_m\}) \cap (\partial S_n^+ - \{0\}) \subset S_{n,n-1}^+$. The next chapter will show that the intersection $(\text{span}\{M_1, \dots, M_m\}) \cap (\partial S_n^+ - \{0\})$ is actually an embedded

submanifold of Sym_n . This will prove to be an important fact. The key to understanding it is transversality.

Definition 3.8.16. Let $X \subset N$ be a submanifold and let $f : M \rightarrow N$. f intersects X **transversally**, or is **transverse** to X (denoted $f \pitchfork X$) if, for all $p \in f^{-1}(X)$,

$$T_{f(p)}N = T_{f(p)}X + f_{*,p}(T_pM).$$

Two submanifolds, X and Y , of M intersect transversally if, for all $p \in X \cap Y$,

$$T_pM = T_pX + T_pY.$$

Lemma 3.8.17. Let $X \subset N$ be a submanifold and let $f : M \rightarrow N$ be a smooth map. If f intersects X transversally, then, $Z = f^{-1}(X)$ is a submanifold of M with $\text{codim}(Z) = \text{codim}(X)$. Moreover, if $p \in Z$, then $T_pZ = (f_{*,p})^{-1}(T_{f(p)}X)$.

Proof. See [16] □

Chapter 4 will show that $\text{span} \{M_1, \dots, M_m\}$ intersects $S_{n,n-1}^+$ transversally when the quadratic map defined by the M_i is round. This will imply that the intersection is an embedded submanifold of Sym_n . This section concludes with a simpler application of Lemma 3.8.17. It will be called on in the next chapter.

Lemma 3.8.18. Let $K \subset \mathbb{R}^n$ be a definite convex cone which is such that $\partial_{\mathbb{R}^n}K - \{o\}$ is a submanifold of \mathbb{R}^n . Let $u \in \mathbb{R}^n$ be an element of $\text{Int}_{\mathbb{R}^n}(K^*)$ (which is nonempty by the definiteness of K). Finally, let $a \in \mathbb{R}$, $a > 0$. Then, H_u^a intersects $\partial_{\mathbb{R}^n}K - \{o\}$ transversally.

Proof. Let $\partial_{\mathbb{R}^n}^\circ K$ denote $\partial_{\mathbb{R}^n}K - \{o\}$ and let p be a point in $H_u^a \cap \partial_{\mathbb{R}^n}K = H_u^a \cap \partial_{\mathbb{R}^n}^\circ K$. Under the usual identification of $T_p\mathbb{R}^n$ with \mathbb{R}^n , $T_pH_u^a$ is identified with u^\perp . To show that $H_u^a \pitchfork \partial_{\mathbb{R}^n}^\circ K$ it therefore suffices to find an element in $T_p\partial_{\mathbb{R}^n}^\circ K$ that is not in u^\perp . That element is p itself (p is in $T_p\partial_{\mathbb{R}^n}^\circ K$ because the curve $p + tp$ is in $\partial_{\mathbb{R}^n}^\circ K$ for sufficiently small $t \in \mathbb{R}$). Moreover, $p \in T_p\mathbb{R}^n$ cannot be contained in $T_pH_u^a = u^\perp$, since $u \in \text{Int}_{\mathbb{R}^n}(K^*)$ implies that $\langle u, p \rangle > 0$. □

Corollary 3.8.19. Let $K \subset \mathbb{R}^n$ be a cone satisfying the conditions of the previous lemma, in addition to one more: $\text{Int}_{\mathbb{R}^n}K \neq \emptyset$. Then, still using the terms of the lemma, $H_u^a \cap (\partial_{\mathbb{R}^n}K - \{o\})$ is diffeomorphic to \mathbb{S}^{n-2} .

Proof. By Lemma 3.8.18, the intersection $H_u^a \cap (\partial_{\mathbb{R}^n} K - \{o\})$ is an embedded submanifold of H_u^a . It, moreover, is a convex body in H_u^a : it is convex, it is compact because K is closed (see Lemma 3.4.18), and lastly, since $\text{Int}_{\mathbb{R}^n} K \neq \emptyset$ by assumption, $\text{Int}_{H_u^a}(H_u^a \cap K - \{o\}) \neq \emptyset$.

Proposition 3.8.3 now implies that the boundary of this convex body, which is just $H_u^a \cap (\partial_{\mathbb{R}^n} K_\psi - \{o\})$, is diffeomorphic to \mathbb{S}^{n-2} (since H_u^a is $n - 1$ -dimensional). ■ □

3.8.5 SARD'S THEOREM

Sard's theorem is an essential part of the proof of the Instrumental Lemma (Lemma 2.2.7). This section presents Sard's theorem, together with a version of it that is specially suited to the proof of the Instrumental Lemma.

Definition 3.8.20. Let M and N be smooth manifolds and let $f : M \rightarrow N$ be a smooth map. A point $p \in M$ is a **regular point** of f if the differential $f_{*,p} : T_p M \rightarrow T_{f(p)} N$ is surjective. Otherwise, p is a **critical point**. A point $q \in N$ is a **critical value** of f if $f^{-1}(q)$ contains at least one critical point. Otherwise, q is a **regular value**.

Theorem 3.8.21. (Sard's Theorem) Let $f : M \rightarrow N$ be a smooth map. The set of critical values of f has measure zero.

Remark 3.8.22. A proof of Sard's theorem can be found in [16], as can an explanation of the notion of a measure zero subset of a manifold.

The specialized version of Sard's theorem that is needed for the Instrumental Lemma follows immediately from Theorem 3.8.21.

Theorem 3.8.23. (Special Sard's Lemma) Let V and L be finite dimensional real vector spaces and let $f : V \rightarrow L$ be a smooth map. If $\dim V < \dim L$, then $\text{Int}_L(\text{Im } f) = \emptyset$.

3.8.6 THE NONEXISTENCE OF CERTAIN RETRACTIONS

The proof of the Roundness Theorem will call on a well-known fact about the nonexistence of retractions from \mathbb{D}^n to \mathbb{S}^{n-1} .

Definition 3.8.24. Let X be a topological space and $A \subset X$. A continuous map $r : X \rightarrow A$ is a **retraction** if $r \circ i_A = I_A$ where $i_A : A \hookrightarrow X$ is the inclusion of A into X and I_A the identity map of A .

Theorem 3.8.25. There is no retraction from \mathbb{D}^n to \mathbb{S}^{n-1} for any $n \geq 1$.

Proof. See [16]. □

Lemma 3.8.26. Let $p \in \text{Int}_{\mathbb{R}^n}(\mathbb{D}^n)$. Define $r_p : \mathbb{D}^n - \{p\} \rightarrow \mathbb{S}^{n-1}$ by $x \mapsto \frac{1}{\|x-p\|}(x-p)$. Then, $r_p \circ i_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$ is a homeomorphism ($i_{\mathbb{S}^{n-1}}$ is the inclusion of \mathbb{S}^{n-1} into \mathbb{D}^n).

Proof. The inverse of $r_p \circ i_{\mathbb{S}^{n-1}}$ is the map taking $v \in \mathbb{S}^{n-1}$ to $p + t(v)v$, where $t(v)$ is the unique positive number such that $\|p + t(v)v\|^2 = 1$,

$$t(v) = -\langle v, p \rangle + \sqrt{\langle v, p \rangle^2 + 4(1 - \|p\|^2)}$$

$t(v)$ depends continuously on v ; so too, therefore, does $p + t(v)v$. □

3.8.7 EMBEDDINGS OF SPHERES

This section establishes some important facts about smooth embeddings of spheres into other spheres. They will be used in the proof of the Roundness Theorem.

SOME TERMINOLOGY

Throughout this section, the sphere \mathbb{S}^m will sometimes be thought of as a subset of \mathbb{S}^n (when $m < n$). On those occasions, \mathbb{S}^m should be thought of as the subset $\mathbb{S}^n \cap \mathbb{R}^{m+1}$, where $\mathbb{R}^{m+1} \subset \mathbb{R}^{n+1}$ is the subspace $\{(x^1, \dots, x^{m+1}, 0, \dots, 0) : x^i \in \mathbb{R}\}$. The inclusion map $i_{\mathbb{S}^m} : \mathbb{S}^m \rightarrow \mathbb{S}^n$ will be

referred to as the *standard embedding* of \mathbb{S}^m ; the subset $\mathbb{S}^m \subset \mathbb{S}^n$ will sometimes be called the *standard copy* of \mathbb{S}^m in \mathbb{S}^n .

If $h : \mathbb{S}^m \rightarrow \mathbb{S}^n$ is a continuous embedding, the image $S = h(\mathbb{S}^m)$ will be said to bound a $m + 1$ -dimensional disc in \mathbb{S}^n if h extends to a continuous embedding $\bar{h} : \mathbb{D}^{m+1} \rightarrow \mathbb{S}^n$ of \mathbb{D}^{m+1} , in which case S will be said to bound the disc $D = \bar{h}(\mathbb{D}^{m+1})$.

EMBEDDINGS OF SPHERES

The Roundness Theorem calls on two facts about smooth embeddings of spheres into larger spheres: the Generalized Schoenflies theorem, and a generalization of it due to Stallings [39]. Both results state that *smooth* embeddings of a sphere \mathbb{S}^m into a larger sphere \mathbb{S}^n , are equivalent, in the sense defined below, to the standard embedding of \mathbb{S}^m in \mathbb{S}^n .

Definition 3.8.27. Two continuous embeddings $h_i : \mathbb{S}^m \rightarrow \mathbb{S}^n$, $i = 1, 2$ are **equivalent** if there exists a homeomorphism $H : \mathbb{S}^n \rightarrow \mathbb{S}^n$ such that $H \circ h_1 = h_2$.

Both the Generalized Schoenflies theorem and Stallings's theorem have a precedent, namely the classical Schoenflies theorem.

Theorem 3.8.28. (Schoenflies) Let $h : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a continuous embedding. Then, h is equivalent to the standard embedding $i_{\mathbb{S}^1} : \mathbb{S}^1 \hookrightarrow \mathbb{S}^2$.

It follows that the closure of each component of $h(\mathbb{S}^1)^c$ is homeomorphic to the closed disk \mathbb{D}^2 . When $n \geq 2$, the analogous statement about continuous embeddings of \mathbb{S}^{n-1} in \mathbb{S}^n is not true; the components of $h(\mathbb{S}^{n-1})^c$ need not be homeomorphic to the closed disk \mathbb{D}^n . The Alexander Horned Sphere (see [19]) is a well known example of an embedding $h_A : \mathbb{S}^2 \rightarrow \mathbb{S}^3$ for which one of the connected components of $h_A(\mathbb{S}^2)^c$ is not homeomorphic to \mathbb{D}^2 . h_A , therefore, is not equivalent to the standard embedding of \mathbb{S}^2 into \mathbb{S}^3 .

Things are more tractable if $h : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^n$ is a *smooth* embedding.

Theorem 3.8.29. (Generalized Schoenflies, [8]) Let $n \geq 0$. Let $h : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^n$ be a smooth embedding. Then, h is equivalent to the standard embedding $i_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$.

Remark 3.8.30. The statment of the Generalized Schoenflies that appears in [8] is slightly different, and more general, than the statement of Theorem 3.8.29. What appears in [8] is this: Let $h : \mathbb{S}^{n-1} \times [0, 1] \rightarrow \mathbb{S}^n$ be a continuous bijection (that is, a continous embedding). Then, the closure of either connected component of $h(\mathbb{S}^{n-1} \times \frac{1}{2})$ is homeomorphic to the closed unit disk \mathbb{D}^n .

This statement implies 3.8.29. For, if $h : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^n$ is a smooth embedding, then $h(\mathbb{S}^{n-1})$ has a uniform tubular neighbourhood, which is to say that h extends to a smooth embedding $H : \mathbb{S}^{n-1} \times [0, 1] \rightarrow \mathbb{S}^n$ for which $H \circ i_{\frac{1}{2}} = h$, where $i_{\frac{1}{2}} : \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n \times [0, 1]$ is the map $x \mapsto (x, \frac{1}{2})$. Theorem 3.8.29 is therefore implied by the version of the Generalized Schoenflies theorem that appears in [8]. ■

The Generalized Schoenflies theorem begs a question: are smooth embeddings of \mathbb{S}^m into \mathbb{S}^n equivalent to the standard embedding $i_{\mathbb{S}^m} : \mathbb{S}^m \hookrightarrow \mathbb{S}^n$ for all $m < n$? Stallings's theorem provides a partial answer.

Theorem 3.8.31. (Stallings, [39]) Let $n \geq 5$ and $m \leq n - 3$. Let $h : \mathbb{S}^m \rightarrow \mathbb{S}^n$ be a smooth embedding. Then, h is equivalent to the standard embedding $i_{\mathbb{S}^m} : \mathbb{S}^m \hookrightarrow \mathbb{S}^n$.

For the purposes of the Roundness Theorem, the most important consequence of Theorems 3.8.29 and 3.8.31 is this: every smooth embedding $h : \mathbb{S}^m \rightarrow \mathbb{S}^n$ (with m either equal to $n - 1$ or satisfying the bounds of Theorem 3.8.31) extends to a continuous embedding $\bar{h} : \mathbb{D}^{m+1} \rightarrow \mathbb{S}^n$. Thus, the image $h(\mathbb{S}^m)$ bounds the continuously embedded disk $\bar{h}(\mathbb{D}^{m+1})$.

4

Quadratic Maps

This chapter introduces quadratic maps in earnest. It lays a foundation for the proofs of the Roundness Theorem and the Instrumental Lemma, both of which appear in the next chapter.

It follows immediately from the definition of quadratic maps that the image of a quadratic map is a cone. The question that drives this chapter and the next is this: when is the image of a quadratic map a *convex* cone?

The chapter begins by redefining quadratic maps. Though they were defined in the introduction to this thesis, it will prove easier to develop a theory of quadratic maps using the more general definition that appears in Section 4.1. Then, the chapter introduces an essential tool for studying quadratic maps: the associated linear map. Every quadratic map can be uniquely associated with a certain linear map that imparts a great wealth of information about the quadratic map to which it is associated. Section 4.2 catalogs a variety of properties of quadratic maps that

can be readily derived from their associated linear maps.

Sections 4.3 to 4.5 update some of the definitions that were made in Chapter 2 to reflect the new definition of quadratic maps. Section 4.3 redefines stable convexity. Section 4.4 redefines definiteness and catalogs a number of important properties of definite quadratic maps. Section 4.4 culminates in a very important description of the convexification of the image of a definite quadratic map. Section 4.5 studies round quadratic maps in detail. Besides establishing the existence of round maps, Section 4.5 uncovers the nature of the boundaries of their images. Finally, Section 4.6 defines a useful equivalence relation on the set of quadratic maps from one space to another.

4.1 QUADRATIC MAPS DEFINED

The introduction to this thesis defined quadratic maps in two equivalent ways: first as maps from \mathbb{R}^n to \mathbb{R}^m whose components are homogeneous, degree 2 polynomials, and then as m -tuples of symmetric matrices, $(M_1, \dots, M_m) \in \text{Sym}_n \times \dots \times \text{Sym}_n$. This section will establish a final, equivalent definition that will be used predominantly from now on.

The forthcoming definition is given in terms of abstract inner product spaces V and W , rather than \mathbb{R}^n and \mathbb{R}^m . This is because there will eventually be a need to restrict quadratic maps, even those defined on \mathbb{R}^n , to certain subspaces of their domains (see the proof of the Instrumental Lemma and that of the forward implication in the Roundness theorem). There will also be a need to think of some quadratic maps as mapping into certain subspaces of their codomains (see Remark 4.2.4). In general, there will be no natural choice of coordinates on these subspaces and therefore no obvious way to identify them with some \mathbb{R}^n .

Besides defining quadratic maps, this section introduces an indispensable tool for studying them. Every quadratic map Q mapping from V to W is associated with a linear map mapping from W to $\text{Sym}(V)$, the space of symmetric linear maps of V . Many properties of Q that would be otherwise difficult to express can be stated succinctly in terms of its associated linear map. Section 4.2 will provide

an abundance of examples of such properties.

Throughout this chapter and the next, V and W will denote real, finite-dimensional inner product spaces. The inner products on V and W will be denoted by \langle, \rangle_V and \langle, \rangle_W respectively. The theory that this chapter develops is independent of the choice of the inner products \langle, \rangle_V and \langle, \rangle_W ; this is explained ahead in Section 4.6.

Definition 4.1.1. A map $Q : V \rightarrow W$ is a **quadratic map** if there exists a symmetric bilinear map $B : V \times V \rightarrow W$ such that $Q = B \circ \Delta$ where $\Delta : V \rightarrow V \times V$ is the diagonal map $v \mapsto (v, v)$.

Definition 4.1.2. The set of all quadratic maps from V to W will be denoted by $\mathcal{Q}(V, W)$. When $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, $\mathcal{Q}(V, W)$ will be denoted by $\mathcal{Q}_{n,m}$. Finally, $\mathcal{Q}_{n,n}$ will be denoted by \mathcal{Q}_n .

Remark 4.1.3. It follows from Definition 4.1.1 that the pre- or post-composition of a quadratic map with a linear map is again a quadratic map. ■

This is a convenient place to record a new definition of convex quadratic maps that reflects Definition 4.1.1. This is a formality; the new definition is essentially unchanged from Definition 2.0.4. The definition of stable convexity arrives in Section 4.3.

Definition 4.1.4. $Q \in \mathcal{Q}(V, W)$ is **convex** if $\text{Im } Q$ is a convex subset of W .

Definition 4.1.5. The subset of convex quadratic maps in $\mathcal{Q}(V, W)$ will be denoted by $\text{Convex}(V, W) \subset \mathcal{Q}(V, W)$ (or $\text{Convex}_{n,m} \subset \mathcal{Q}_{n,m}$ if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, and Convex_n if $V = W = \mathbb{R}^n$).

Remark 4.1.6. The new definition of quadratic maps, Definition 4.1.1, generalizes the old one, Definition 1.2.3: if $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a quadratic map defined, in the terms of Definition 1.2.3, by the symmetric matrices $M_1, \dots, M_m \in \text{Sym}_n$, then $Q = B \circ \Delta$ where $\Delta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the diagonal map and $B : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the symmetric bilinear map given by

$$(x, y) \mapsto \begin{pmatrix} \langle x, M_1 y \rangle \\ \vdots \\ \langle x, M_m y \rangle \end{pmatrix}$$

The terminology of Definition 1.2.3 will still occasionally be used when discussing quadratic maps from \mathbb{R}^n to \mathbb{R}^m . ■

If $Q = B \circ V \in \mathcal{Q}(V, W)$ is a quadratic map defined by the bilinear, symmetric map B , then Q and B satisfy a polarization identity: $B(x, y) = \frac{1}{2} [Q(x + y) - Q(x) - Q(y)]$. It follows that B is the *only* bilinear, symmetric map that defines Q . As a result, $\mathcal{Q}(V, W)$ is in bijective correspondence with the set of bilinear symmetric maps from $V \times V$ to W .

Let $u \in W$ and let $B : V \times V \rightarrow W$ be a bilinear, symmetric map. Then, $(x, y) \mapsto \langle B(x, y), u \rangle_V$ is a bilinear symmetric *form* on V . Every bilinear symmetric form on V can be written as $(x, y) \mapsto \langle x, My \rangle_V$ for some unique $M \in \text{Sym}(V)$. Thus,

$$\langle B(x, y), u \rangle_W = \langle x, M_u y \rangle_V \quad \forall x, y \in V \quad (4.1.1)$$

for some unique $M_u \in \text{Sym}(V)$. The relation (4.1.1) associates each $u \in W$ with an $M_u \in \text{Sym}_n$. The association, moreover, is linear. Thus, to each bilinear symmetric map $B : V \times V \rightarrow W$, there corresponds a *linear* map $\psi_B : W \rightarrow \text{Sym}(V)$ that is defined by equation (4.1.1). Conversely, given a linear map from W to $\text{Sym}(V)$, equation (4.1.1) defines a bilinear map from $V \times V$ to W . Equation (4.1.1) therefore defines a bijective correspondence between $\mathcal{L}(W, \text{Sym}(V))$ and the set of symmetric bilinear maps from $V \times V$ to W .

It now follows that there is a bijective correspondence between the set of linear maps $\mathcal{L}(W, \text{Sym}(V))$ and the set of quadratic maps $\mathcal{Q}(V, W)$, the correspondence $Q \in \mathcal{Q}(V, W) \leftrightarrow \psi \in \mathcal{L}(W, \text{Sym}(V))$ being defined by the

equation

$$\langle Q(x), u \rangle_W = \langle x, \psi(u)x \rangle_V \quad \forall x \in V, u \in W \quad (4.1.2)$$

Equation (4.1.2) is the most important equation in this chapter. Henceforth, it will be referred to as the *fundamental relation*.

Defintion 4.1.7. Let $Q \in \mathcal{Q}(V, W)$ be a quadratic map. The unique linear map $\psi : W \rightarrow \text{Sym}(V)$ satisfying equation (4.1.2) is the **linear map associated with Q** , or simply the **associated linear map** if Q is understood; this will sometimes be abbreviated as ALM.

Defintion 4.1.8. If $\psi \in \mathcal{L}(W, \text{Sym}(V))$, the unique quadratic map $Q : V \rightarrow W$ satisfying equation (4.1.2) is the **quadratic map associated with ψ** , or simply the **associated quadratic map** when ψ is understood; this will sometimes be abbreviated as AQM.

Example 4.1.9. The associated linear map, $\psi : \mathbb{R}^m \rightarrow \text{Sym}_n$ of a quadratic map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that is defined by $M_1, \dots, M_m \in \text{Sym}_n$ is especially easy to describe: ψ maps the i^{th} basis vector of \mathbb{R}^m to M_i . This follows immediately from the fundamental relation (4.1.2).

Associated linear maps provide an extremely effective means for studying quadratic maps. The following section will make this clear.

It will be useful later on to have a symbol for the bijective correspondence induced by equation (4.1.2).

Defintion 4.1.10. The bijection induced by equation (4.1.2) will be denoted by $\rho : \mathcal{Q}(V, W) \rightarrow \mathcal{L}(W, \text{Sym}(V))$.

4.2 BASIC PROPERTIES OF QUADRATIC MAPS

The fundamental relation, equation (4.1.2), contains a trove of information about its quadratic map, Q . It also allows that information to be expressed neatly in

terms of ψ , the ALM of Q , and objects associated with ψ (its kernel, for example; see Lemma 4.2.2). What follows below is a collection of properties of quadratic maps, all expressed in terms of their ALMs. Each property derives from and exemplifies the usefulness of the fundamental relation (4.1.2).

Throughout this section, Q will denote a quadratic map in $\mathcal{Q}(V, W)$ and $\psi \in \mathcal{L}(W, \text{Sym}(V))$ will denote its associated linear map.

Remark 4.2.1. The image of Q has a couple of properties that are too basic to warrant their own subsections. First, the image of Q is necessarily a cone in W . And second, if V , the domain of Q , is 1-dimensional, the image of Q is a half-line in W : $\text{Im } Q = \{\lambda Q(v) : \lambda \geq 0\}$ where $v \in V$ is any fixed, nonzero element of V . ■

4.2.1 THE SMALLEST SUBSPACE CONTAINING $\text{Im } Q$

Proposition 4.2.2. $(\text{Im } Q)^\perp = \ker \psi$.

Proof. If $u \in \ker \psi$, then $\langle x, \psi(u)x \rangle = 0$ for all $x \in V$. The fundamental relation (4.1.2) then implies that $\langle Q(x), u \rangle = 0$ for all $x \in V$. Thus, $u \in (\text{Im } Q)^\perp$, and $\ker \psi \subset (\text{Im } Q)^\perp$. The opposite containment follows from the fact that if $M \in \text{Sym}(V)$, $\langle x, Mx \rangle = 0$ for all $x \in V$ if and only if $M = 0$. □

Corollary 4.2.3. $(\ker \psi)^\perp$ is the smallest subspace containing $\text{Im } Q$. $(\ker \psi)^\perp$ contains $\text{Conv}(\text{Im } Q)$.

Proof. If $S \subset W$ is any subset of W , $S^{\perp\perp}$ is the smallest subset containing S . For the second part of the corollary, note that the convexification of any subset $S \subset W$ is contained in any subspace of W that contains S . □

Remark 4.2.4. Proposition 4.2.2 implies that every quadratic map $Q \in \mathcal{Q}(V, W)$ can be written as $Q = i \circ \bar{Q}$ where $i : (\ker \psi)^\perp \rightarrow W$ is the inclusion map and $\bar{Q} : V \rightarrow (\ker \psi)^\perp$ is just Q thought of as mapping into $(\ker \psi)^\perp$ (or, $\bar{Q} = \pi \circ Q$, where $\pi : V \rightarrow (\ker \psi)^\perp$ is the orthogonal projection of V onto $\ker \psi$). This underscores the need to have defined quadratic maps between abstract vector spaces, since $(\ker \psi)^\perp$ may not have a natural choice of coordinates. ■

4.2.2 THE INTERIOR OF $\text{Conv}(\text{Im } Q)$

The following proposition plays an important role in the proof of the Instrumental Lemma (Lemma 2.2.7).

Proposition 4.2.5. $\text{Int}_W(\text{Conv}(\text{Im } Q)) = \emptyset$ if and only if $\ker \psi \neq \{o\}$.

Proof. The backward implication $\ker \psi \neq \{o\} \Rightarrow \text{Int}_W(\text{Conv}(\text{Im } Q)) = \emptyset$ follows from Corollary 4.2.3: $\text{Conv}(\text{Im } Q)$ cannot have a nonempty interior if $\ker \psi \neq \{o\}$, since $\text{Conv}(\text{Im } Q) \subset (\ker \psi)^\perp$.

If, conversely, $\text{Int}_W(\text{Conv}(\text{Im } Q)) = \emptyset$, then, by Proposition 3.4.6, there exists a proper subspace $P \subset W$ containing $\text{Conv}(\text{Im } Q)$. By Corollary 4.2.3, $(\ker \psi)^\perp$ is contained in P . Thus, $o \neq P^\perp \subset \ker \psi$. \square

It is similarly true that if $\ker \psi \neq \{o\}$, then the interior of the image $\text{Im } Q$ itself is empty. However, the converse fails. As the proof of the Instrumental Lemma will show, there are many quadratic maps Q whose image has an empty interior, $\text{Int}_W(\text{Im } Q) = \emptyset$, but whose ALM ψ has a trivial kernel, $\ker \psi = o$.

Example 4.2.6. Let $Q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the quadratic map given by

$$Q : x \in \mathbb{R}^2 \mapsto \begin{pmatrix} (x^1)^2 - (x^2)^2 \\ 2x^1x^2 \\ (x^1)^2 + (x^2)^2 \end{pmatrix}$$

The image of Q has no interior in \mathbb{R}^3 . Yet, $\ker \psi = \{o\}$, where ψ is the ALM of Q . Note that the interior of the convexification of $\text{Im } Q$ is not empty.

4.2.3 THE PREIMAGE OF $o \in W$

The next proposition states that the preimage $Q^{-1}(o)$ can be computed by solving a homogeneous system of multivariate quadratic polynomials, namely $\langle x, M_1 x \rangle_V = \dots = \langle x, M_{\dim \text{Im } \psi} x \rangle_V = o$, where $\{M_1, \dots, M_{\dim \text{Im } \psi}\}$ is a basis of $\text{Im } \psi$. The proposition uses the notation \mathcal{Z}_C established in Definition 3.5.6.

Proposition 4.2.7. $Q^{-1}(0) = \mathcal{Z}_{\text{Im } \psi}$.

Proof. If $Q(x) = 0$, then $\langle Q(x), u \rangle = 0$ for all $u \in W$. But then equation (4.1.2) implies that $\langle x, \psi(u)x \rangle = 0$ for all $u \in W$. Thus, $x \in \mathcal{Z}_{\text{Im } \psi}$. The opposite containment $\mathcal{Z}_{\text{Im } \psi} \subset Q^{-1}(0)$ can be obtained in a very similar way. \square

Thus, the preimage of 0 under Q is nonzero if and only if the subspace $\text{Im } \psi \subset \text{Sym}(V)$ is a nice subspace (recall Definition 3.5.1).

4.2.4 THE CLOSEDNESS OF $\text{Im } Q$

Proposition 4.2.8. $\text{Im } Q$ is closed in W if $Q^{-1}(0) = 0$.

Proof. This follows immediately from Proposition 3.7.4 (in the terminology of Section 3.7, quadratic maps are 2-homogeneous). \square

4.2.5 THE PREIMAGE OF A LINE IN W

Proposition 4.2.9. Let $l \subset W$ be a 1-dimensional subspace. Then,

$$Q^{-1}(l) = \mathcal{Z}_{\psi(l^\perp)}.$$

Proof. If $x \in Q^{-1}(l)$, then $\langle Q(x), u \rangle = 0$ for all $u \in l^\perp$. By equation (4.1.2), this implies that $\langle x, \psi(u)x \rangle = 0$ for all $u \in l^\perp$, and therefore that $x \in \mathcal{Z}_{\psi(l^\perp)}$. Thus, $Q^{-1}(l) \subset \mathcal{Z}_{\psi(l^\perp)}$.

If $x \in \mathcal{Z}_{\psi(l^\perp)}$, then $\langle x, \psi(u)x \rangle = 0$ for all $u \in l^\perp$. Again by equation (4.1.2), it follows that $\langle Q(x), u \rangle = 0$ for all $u \in l^\perp$. Thus, $Q(x) \in (l^\perp)^\perp = l$. \square

Like Proposition 4.2.7, Proposition 4.2.9 says that a preimage under Q of a line $l \subset W$, can be computed by solving a system of quadratic polynomials.

Proposition 4.2.9 can produce a necessary condition for the preimage of a point p in W to be nonempty. For, if $l \subset W$ is the 1-dimensional subspace spanned by p , then, $Q^{-1}(p) \neq \emptyset$ only if $Q^{-1}(l) \neq \emptyset$. Proposition 4.2.9 now implies this,

Corollary 4.2.10. *Let $p \in W$ be a point. Then,*

$$p \in \text{Im } Q \Rightarrow \mathcal{Z}_{\psi(p^\perp)} \neq \{0\}$$

Equivalently,

$$p \in \text{Im } Q \Rightarrow \psi(p^\perp) \text{ is a nice subspace of } \text{Sym}(V)$$

The necessary condition in the corollary is *not* sufficient. For, even if $\mathcal{Z}_{\psi(p^\perp)} \neq \{0\}$, one of two possibilities may thwart the conclusion that $Q^{-1}(p) \neq \emptyset$: Q may map onto $-p$, but not p , or Q may map all of $\mathcal{Z}_{\psi(p^\perp)}$ onto $0 \in W$. The following examples exhibit each of these possibilities.

Example 4.2.11. *Consider the standard quadratic map in n dimensions. The basis vector $-e_1$ satisfies $\mathcal{Z}_{\psi(-e_1^\perp)} \neq \{0\}$, but $-e_1 \notin \text{Im } Q$.*

Example 4.2.12. *Consider the quadratic map $Q : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by*

$$x \in \mathbb{R}^3 \mapsto \begin{pmatrix} (x^1)^2 - (x^2)^2 \\ 2x^1x^2 \\ (x^1)^2 + (x^2)^2 \end{pmatrix}$$

If $l \subset \mathbb{R}^3$ is the e_3 -axis, then $Q^{-1}(l) = l \neq \{0\}$, but $e_3 \notin \text{Im } Q$. Note that Q maps every point in $Q^{-1}(0)$ to $0 \in \mathbb{R}^3$.

4.2.6 THE DUAL OF $\text{Im } Q$

Chapter 3 cemented the role of dual cones in the study of convexity. It showed that, under certain conditions, the convexification of a cone is equal to the cone's double dual. Accordingly, the next proposition is fundamentally important for the study of the convexity of $\text{Im } Q$. In words, the proposition says that the dual of $\text{Im } Q$ is the preimage under ψ of the cone of positive semidefinite maps in $\text{Sym}(V)$.

Proposition 4.2.13. $(\text{Im } Q)^* = \psi^{-1}(S^+(V)).$

Proof. $u \in W$ is in $(\text{Im } Q)^*$ if and only if $\langle u, Q(x) \rangle \geq 0$ for all $x \in V$. But then, by the fundamental relation (4.1.2), $\langle x, \psi(u)x \rangle \geq 0$ for all $x \in V$. Thus, $u \in \psi^{-1}(S^+(V))$. The containment $\psi^{-1}(S^+(V)) \subset (\text{Im } Q)^*$ follows in the same manner. \square

It follows that the double dual $(\text{Im } Q)^{**}$ of $\text{Im } Q$ is equal to $(\psi^{-1}(S^+(V)))^*$. This, with the help of the results of Chapter 3 (in particular, Theorem 3.4.36), will lead to a description of $\text{Conv}(\text{Im } Q)$ for a large class of quadratic maps (namely, definite ones). This is articulated in Section 4.4, which studies definite quadratic maps in detail.

The importance of $\psi^{-1}(S^+(V))$ in the remainder of the paper warrants new notation.

Definition 4.2.14. Let $\psi \in \mathcal{L}(W, \text{Sym}(V))$. The pull-back cone of $S^+(V)$ by ψ will be denoted by K_ψ .

Remark 4.2.15. Note that $\ker \psi \subset K_\psi$. \blacksquare

4.2.7 THE PREIMAGE OF A HYPERPLANE IN W

This section's proposition describes the preimage under Q of a hyperplane in W ; the proposition will be used later on to describe the preimages of faces of $\text{Im } Q$. Whereas the preimages of $0 \in W$, and of a line $l \subset W$ are solution sets to *systems* of multivariate quadratic polynomials, the preimage of a hyperplane in W is the solution set to a *single* polynomial.

Proposition 4.2.16. Let $u \in W$. Then, $Q^{-1}(u^\perp) = \mathcal{Z}_{\psi(u)}$.

Proof. If $x \in \mathcal{Z}_{\psi(u)}$, then $\langle x, \psi(u)x \rangle = 0$. By equation (4.1.2), $\langle Q(x), u \rangle = 0$; x is therefore in $Q^{-1}(u^\perp)$. This proves the containment $\mathcal{Z}_{\psi(u)} \subset Q^{-1}(u^\perp)$. The opposite containment follows from the fundamental relation (4.1.2) in the same way. \square

Proposition 4.2.16 has a corollary that plays a very important role in the understanding the boundary of $\text{Conv}(\text{Im } Q)$. The corollary reduces the

computation of $Q^{-1}(u^\perp)$ to the computation of the kernel of a linear map, when $u \in K_\psi$.

Corollary 4.2.17. *Let $u \in K_\psi$. Then, $Q^{-1}(u^\perp) = \ker \psi(u)$.*

Proof. Lemma 4.2.16 says that $Q^{-1}(u^\perp) = \mathcal{Z}_{\psi(u)}$. But, if $M \in S^+(V)$, then $\mathcal{Z}_M = \ker M$. Thus, $\mathcal{Z}_{\psi(u)} = \ker \psi(u)$. \square

Remark 4.2.18. Corollary 4.2.17 says that when $u \in K_\psi$, the restriction of Q to $\ker \psi(u)$ maps into u^\perp . Let $\bar{Q} : \ker \psi(u) \rightarrow u^\perp$ denote this restriction. Then, the Corollary says that $\text{Im } Q \cap u^\perp$ is the image of the quadratic map \bar{Q} .

Part of what makes the foregoing interpretation of $\text{Im } Q \cap u^\perp$ work is the fact that $Q^{-1}(u^\perp)$ is a *subspace*. It is true, by Lemma 4.2.16, that for any $u \in W$, $\text{Im } Q \cap u^\perp$ is the image of the restriction of Q to $\mathcal{Z}_{\psi(u)}$. However, in general, this restriction cannot be thought of as a quadratic map, simply because $Q^{-1}(u^\perp)$, though a cone in V , is not necessarily a subspace of V (by definition, quadratic maps map from one *vector space* to another). If $Q^{-1}(u^\perp)$ is not a subspace of V , the restriction of Q to $Q^{-1}(u^\perp)$ cannot be thought of as a quadratic map.

The property of $Q^{-1}(u^\perp)$ being a subspace is unique to those $u \in K_\psi$. This is because, for $M \in \text{Sym}(V)$, \mathcal{Z}_M is a subspace of V if and only if $M \in S^+(V)$. ■

4.2.8 FACES OF $\text{Im } Q$

By definition, a face of $\text{Im } Q$ is a subset of the form $u^\perp \cap \text{Im } Q$ where u is a nonzero element of $\partial_W(\text{Im } Q)^*$. Weaving together the conclusions of the last two sections yields this,

Proposition 4.2.19. *Every face of $\text{Im } Q$ is of the form $Q(\ker \psi(u))$ for some nonzero $u \in \partial_W K_\psi$.*

Proof. Because $(\text{Im } Q)^* = K_\psi$, every face of $(\text{Im } Q)$ is of the form $u^\perp \cap \text{Im } Q$ for some nonzero $u \in \partial_W K_\psi$. But, when $u \in K_\psi$, $u^\perp \cap \text{Im } Q = Q(Q^{-1}(u^\perp \cap \text{Im } Q)) = Q(\ker \psi(u))$, by Corollary 4.2.17. \square

4.3 STABLE CONVEXITY AND A TOPOLOGY FOR $\mathcal{Q}(V, W)$

The definitions of Definite, Round, Convex and Stably Convex quadratic maps that were given in Chapter 2 must be harmonized with the new, coordinate-independent definition of quadratic maps that appeared in Section 4.1. This section redefines stable convexity.

The original definition of quadratic maps, Definition 1.2.3, implicitly identified the set of quadratic maps from \mathbb{R}^n to \mathbb{R}^m with the m -fold product $\text{Sym}_n \times \cdots \times \text{Sym}_n$. In the terms of that identification, the original definition of stable convexity, Definition 2.2.1, said this: a quadratic map is stably convex if it is in the interior of the subset of convex quadratic maps in $\text{Sym}_n \times \cdots \times \text{Sym}_n$. Generalizing Definition 2.2.1 thus requires that $\mathcal{Q}(V, W)$ be endowed with a topology. This prerequisite will be dispatched in a moment; for the purposes of the following Definition, it is simply assumed that there is a topology on $\mathcal{Q}(V, W)$.

Definition 4.3.1. A quadratic map $Q \in \mathcal{Q}(V, W)$ is **stably convex** if it is in the interior of the subset of convex maps: $Q \in \text{Int}_{\mathcal{Q}(V, W)}(\text{Convex}(V, W))$.

Because $\mathcal{Q}(V, W)$ is a set of continuous maps from V to W , a suitable topology for it is the compact-open topology (more precisely, the subspace topology that it inherits from the compact-open topology on the set $\mathcal{C}(V, W)$ of all continuous maps from V to W). However, there is a way of looking at this topology that makes it easier to work with.

Recall that the fundamental relation (4.1.2) defines a bijection $\rho : \mathcal{Q}(V, W) \rightarrow \mathcal{L}(W, \text{Sym}(V))$ (see Definition 4.1.10). This bijection, together with the vector space topology on $\mathcal{L}(W, \text{Sym}(V))$, induces a topology on $\mathcal{Q}(V, W)$. According to the following proposition, the topology so induced is the same as the compact-open subspace topology on $\mathcal{Q}(V, W)$.

Proposition 4.3.2. The compact-open topology on $\mathcal{Q}(V, W)$ coincides with the topology induced by the bijection ρ .

Proof. The proof of Proposition 4.3.2 is deferred to Appendix 4.7. □

Lemma 4.3.2 affords a great advantage: arguments about the topology of various subsets of $\mathcal{Q}(V, W)$ can now be made in terms of objects associated with linear maps. This will simplify proofs that would otherwise be cumbersome (see Proposition 4.5.9 ahead, for example).

4.4 DEFINITE QUADRATIC MAPS

This section undertakes a comprehensive study of definite quadratic maps. The study culminates in an important description of the convexification of the image of a definite map (see Proposition 4.4.6).

The Roundness theorem and the Instrumental Lemma both apply only to definite quadratic maps. For this reason, this thesis focuses squarely on definite quadratic maps; indefinite and semidefinite maps are mentioned briefly in Section 7.2.

4.4.1 DEFINITE QUADRATIC MAPS DEFINED

This section reconciles the definition of definite maps given in Chapter 2 with the new definition of quadratic maps, Definition 4.1.1.

When they were first introduced in Chapter 2, definite quadratic maps were defined as those $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that are defined by a tuple $(M_1, \dots, M_m) \in \text{Sym}_n \times \dots \times \text{Sym}_n$ for which the linear span of the M_i is a definite subspace of Sym_n . As was observed earlier, the associated linear map of Q is the linear map $\psi : \mathbb{R}^m \rightarrow \text{Sym}_n$ taking e_i , the i^{th} standard basis vector of \mathbb{R}^m , to M_i . It follows that the linear span of the M_i is the image of ψ . The following definition agrees, then, with the definition of definiteness given in Chapter 4.1.

Definition 4.4.1. *Q is a **definite quadratic map** if $\text{Im } \psi$ is a definite subspace of $\text{Sym}(V)$. Equivalently, Q is definite if $\text{Im } \psi \cap \mathring{S}^+(V) \neq \emptyset$, or if there exists a $u \in W$ such that $\psi(u) \in \mathring{S}^+(V)$.*

Definition 4.4.2. *The set of definite quadratic maps in $\mathcal{Q}(V, W)$ (respectively, $\mathcal{Q}_{n,m}$) is denoted by $\text{Def}(V, W)$ (respectively, $\text{Def}_{n,m}$).*

4.4.2 BASIC PROPERTIES OF DEFINITE MAPS

Let $Q \in \mathcal{Q}(V, W)$ be a definite map and, as usual, let $\psi \in \mathcal{L}(W, \text{Sym}(V))$ be its ALM. The principal result in this section is that the image of Q is a definite cone in W (this is why Q is called definite). This observation, together with the convexity results of Chapter 3 (Theorem 3.4.36 and Lemma 4.2.13), will lead to a trove of information about the convexification of $\text{Im } Q$.

Showing that the image of a definite quadratic map $Q \in \mathcal{Q}(V, W)$ is a definite cone in W requires two things: the image must be shown to be closed and to satisfy $\text{Int}_W(\text{Im } Q)^* \neq \emptyset$. These steps are taken in succession.

Proposition 4.2.8 in Section 4.2 says that the image of Q is closed so long as $Q^{-1}(o) = o$. Proposition 4.2.7 in that same section says that the preimage of $o \in W$ under Q is equal to the set of $x \in V$ that satisfy $\langle x, \psi(u)x \rangle_V = o$ for all $u \in W$. When Q is definite, there exists a $u \in W$ for which $\psi(u)$ is positive definite and for which the only solution to $\langle x, \psi(u)x \rangle_V = o$ is therefore $o \in V$. It follows that $Q^{-1}(o) = o$, which in turn proves the following proposition.

Proposition 4.4.3. *If $Q \in \mathcal{Q}(V, W)$ is definite, then $\text{Im } Q$ is a closed subset of W .*

To take the final step towards showing that $\text{Im } Q$ is a definite cone, note that $\mathring{S}^+(V)$ is an open subset of $\text{Sym}(V)$. Its preimage $\psi^{-1}(\mathring{S}^+(V))$ under ψ is thereby open in W (it is nonempty, precisely because Q is definite). It follows at once that $\text{Int}_W K_\psi \neq \emptyset$. But, by Proposition 4.2.13, K_ψ is just $(\text{Im } Q)^*$. Thus, $\text{Int}_W(\text{Im } Q)^* \neq \emptyset$.

Proposition 4.4.4. *Let $Q \in \mathcal{Q}(V, W)$ be a quadratic map. If Q is definite, then $\text{Im } Q$ is a definite cone in W .*

Corollary 4.4.5. *Let Q be a definite map. Then, $\text{Conv}(\text{Im } Q)$ is a definite cone.*

4.4.3 THE CONVEXITY OF DEFINITE QUADRATIC MAPS

The image of a definite quadratic map $Q \in \mathcal{Q}(V, W)$ is a definite cone in W . All of the results in Chapter 3 on the convexifications of definite cones can now be

brought to bear on the study of $\text{Conv}(\text{Im } Q)$. The relevant results are all contained in Theorem 3.4.36. As usual, throughout this section Q will denote an element of $\mathcal{Q}(V, W)$, and ψ will denote the associated linear map of Q .

Proposition 4.4.6. *If Q is definite, then $\text{Conv}(\text{Im } Q) = K_\psi^*$.*

Proof. When Q is definite, $\text{Im } Q$ is a definite cone. Theorem 3.4.36 says that if $K \subset W$ is a definite cone, then $\text{Conv}(K) = K^{**}$. By Proposition 4.2.13, $(\text{Im } Q)^* = K_\psi$. This proves the proposition. \square

Remark 4.4.7. There is a description of $\text{Conv}(\text{Im } Q)$ that is simpler than the one given in Proposition 4.4.6, and which applies to all quadratic maps Q , not just definite ones. That description, which is discussed in Section 7.3, has a shortcoming: it does not offer up information about $\text{Conv}(\text{Im } Q)$ as readily as does Proposition 4.4.6. The following proposition is an example of the type of insight Proposition 4.4.6 can provide. ■

Proposition 4.4.8. *Let $p \in W$. Then, if Q is definite,*

- $p \in \text{Conv}(\text{Im } Q)$ if and only if $\langle p, u \rangle_W > 0$ for all $u \in \text{Int}_W K_\psi$,
- $p \in \text{Int}_W(\text{Conv}(\text{Im } Q))$ if and only if $\langle p, u \rangle_W > 0$ for all nonzero $u \in K_\psi$,
- $p \in \partial_W(\text{Conv}(\text{Im } Q))$ if and only if $p^\perp \cap K_\psi$ is a face of K_ψ .

Proof. This is Theorem 3.4.36 applied to $\text{Im } Q$, with K_ψ playing the role of $(\text{Im } Q)^*$. \square

The next three corollaries put each of the statements in Proposition 4.4.8 into more fruitful forms.

Corollary 4.4.9. *Let Q be a definite map, and let $p \in W$. Then, $p \in \text{Conv}(\text{Im } Q)$ if and only if $\psi(p^\perp)$ is **not** a definite subspace of $\text{Sym}(V)$ and $\langle p, u \rangle_W \geq 0$ for some $u \in K_\psi$.*

Proof. Suppose that $p \in \text{Conv}(\text{Im } Q)$ and consider the first bulleted point in Proposition 4.4.8. The condition $\langle p, u \rangle_W > 0$ for all $u \in \text{Int}_W K_\psi$ implies that $p^\perp \cap \text{Int}_W K_\psi = \emptyset$, from which it follows that $\psi(p^\perp)$ does not intersect $\overset{\circ}{S}^+(V)$. Thus,

$$p \in \text{Conv}(\text{Im } Q) \Rightarrow \psi(p^\perp) \text{ is not a definite subspace of } \text{Sym}(V) \quad (4.4.1)$$

Moreover, $p \in \text{Conv}(\text{Im } Q)$ implies that $\langle p, u \rangle_W \geq 0$ for every $u \in K_\psi$. This proves the forward implication.

Suppose now that $\psi(p^\perp)$ is not a definite subspace of $\text{Sym}(V)$. It follows that $p^\perp \cap \text{Int}_W K_\psi = \emptyset$. Since $\text{Int}_W K_\psi$ is connected, $\langle p, u' \rangle_W$ must then have the same sign for every $u' \in \text{Int}_W K_\psi$.

The cone K_ψ is equal to the interior of its closure. Thus, if $\langle p, u \rangle_W \geq 0$ for some $u \in K_\psi$, then $\langle p, u' \rangle_W$ must be *positive* for every $u' \in \text{Int}_W K_\psi$. By Proposition 4.4.8, p must then lie in $\text{Conv}(\text{Im } Q)$. \square

The proofs of the remaining corollaries are similar enough to the preceding one to be omitted.

Corollary 4.4.10. *Let Q be a definite map, and let $p \in W$. Then, $p \in \text{Int}_W(\text{Conv}(\text{Im } Q))$ if and only if $\psi(p^\perp)$ is an indefinite subspace of $\text{Sym}(V)$ and $\langle p, u \rangle_W \geq 0$ for some $u \in K_\psi$.*

Corollary 4.4.11. *Let Q be a definite map, and let $p \in W$. Then, $p \in \partial_W(\text{Conv}(\text{Im } Q))$ if and only if $\psi(p^\perp)$ is a semidefinite subspace of $\text{Sym}(V)$ and $\langle p, u \rangle_W \geq 0$ for some $u \in K_\psi$.*

Recall the implication of Corollary 4.2.10,

$$p \in \text{Im } Q \Rightarrow \psi(p^\perp) \text{ is a nice subspace of } \text{Sym}(V) \quad (4.4.2)$$

Suppose now that $p \in W$ is a point for which $\psi(p^\perp)$ is neither a nice nor a definite subspace of $\text{Sym}(V)$. Because $\psi(p^\perp)$ is not definite, either p or $-p$ is in $\text{Conv}(\text{Im } Q)$; this is due to Corollary 4.4.9. However, because $\psi(p^\perp)$ is not nice,

neither p nor $-p$ is in $\text{Im } Q$; this is due to Corollary 4.2.10. Thus, in this case, $\text{Conv}(\text{Im } Q) \neq \text{Im } Q$; Q is not convex. This argument yields the following important convexity theorem.

Theorem 4.4.12. *Suppose that Q is definite. Q is convex if and only if the following equality holds,*

$$\{p \in W \mid \psi(p^\perp) \text{ is not nice}\} = \{p \in W \mid \psi(p^\perp) \text{ is definite}\}$$

Corollary 4.4.13. *If there exists a $p \in W$ such that $\psi(p^\perp)$ is either indefinite or semidefinite but not nice, then Q is not convex and either p or $-p$ lies in the difference $\text{Conv}(\text{Im } Q) - \text{Im } Q$.*

Examples 3.5.12 and 3.5.13 provided examples of indefinite and semidefinite subspaces Sym_n that fail to be nice. They can be used to construct examples of nonconvex quadratic maps.

Example 4.4.14. *Let M_1, M_2 and M_3 be defined as they were in Example 3.5.12 and let $M_4 = I_4$ (the identity map on \mathbb{R}^4). By Corollary 4.4.9 the quadratic map Q defined by the M_i is not convex. To see this, note that by Theorem 4.4.12, $e_4 \in \mathbb{R}^4$ is in the convexification of $\text{Im } Q$ since $\psi(e_4^\perp) = \text{span}\{M_1, M_2, M_3\}$ is indefinite (where, as usual, $\psi : \mathbb{R}^4 \rightarrow \text{Sym}_4$ is the ALM of Q). Explicitly,*

$$e_4 = \frac{1}{2} [Q(x_1) + Q(x_2)]$$

where

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

However, e_4 is not in $\text{Im } Q$ since $\psi(e_4^\perp)$ is not a nice subspace. Q , therefore, is not convex, by Theorem 4.4.12.

Theorem 4.4.12, together with Calabi's Niceness Theorem (Theorem 3.5.9), proves Calabi's Convexity Theorem 2.1.5. For, under the hypotheses of Calabi's Convexity Theorem, the equality

$$\{p \in \mathbb{R}^3 \mid \psi(p^\perp) \text{ is not nice}\} = \{p \in \mathbb{R}^3 \mid \psi(p^\perp) \text{ is definite}\}$$

that is required by Theorem 4.4.12 is assured by Calabi's Niceness Theorem (each $\psi(p^\perp)$ is either a 1-dimensional or 2-dimensional subspace of Sym_n and therefore not nice if and only if it is definite).

Section 3.5 stated that the failure of Calabi's Niceness Theorem to always hold accounts for the existence of nonconvex quadratic maps in $\mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ when $m \geq 4$. This statement is affirmed by Theorem 4.4.12.

Finally, again in conjunction with Calabi's Niceness Theorem, Theorem 4.4.12 also implies Dines' theorem (Theorem 2.1.1). To see this, let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^2$ be a quadratic map, and let $Q' : \mathbb{R}^n \rightarrow \mathbb{R}^3$ be the augmented map,

$$Q' \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} Q(x) \\ ||x||^2 \end{pmatrix}$$

Q' is definite and, by Calabi's Convexity Theorem, convex. But, identifying \mathbb{R}^2 with $e_3^\perp \subset \mathbb{R}^3$, $\text{Im } Q$ is just the image of $\text{Im } Q'$ under the orthogonal projection of \mathbb{R}^3 onto e_3^\perp . It follows that $\text{Im } Q$ is convex.

4.4.4 THE BOUNDARY OF $\text{Conv}(\text{Im } Q)$ WHEN Q IS DEFINITE

This section establishes a couple of facts about the boundary of $\text{Conv}(\text{Im } Q)$ when $Q \in \mathcal{Q}(V, W)$ is a definite map. The first fact, Proposition 4.4.16, describes the preimage under Q of the boundary $\partial_W \text{Conv}(\text{Im } Q)$; it plays an important role in the proof of the Roundness Theorem.

Proposition 4.4.16 is an immediate consequence of the following lemma, which says that the faces of $\text{Conv}(\text{Im } Q)$ are the sets $\text{Conv}(Q(\ker \psi(u)))$ as u

varies over $\partial_W K_\psi - \{0\}$.

Lemma 4.4.15. *Let Q be a definite map. Then,*

$$\begin{aligned}\partial_W(\text{Conv}(\text{Im } Q)) &= \bigcup_{u \in \partial_W K_\psi, u \neq 0} u^\perp \cap \text{Conv}(\text{Im } Q) \\ &= \bigcup_{u \in \partial_W K_\psi, u \neq 0} \text{Conv}(u^\perp \cap \text{Im } Q) \\ &= \bigcup_{u \in \partial_W K_\psi, u \neq 0} \text{Conv}(Q(\ker \psi(u)))\end{aligned}$$

Proof. Proposition 3.4.39 says that when $K \subset W$ is a definite cone,

$$\partial_W(\text{Conv}(K)) = \bigcup_{u \in \partial_W K^*, u \neq 0} \text{Conv}(K \cap u^\perp)$$

The first equality above follows from this, and from $(\text{Im } Q)^* = K_\psi$. By Proposition 4.2.19, when $u \in \partial_W K_\psi$, $u^\perp \cap \text{Im } Q = Q(\ker \psi(u))$; this proves the third equality. \square

Proposition 4.4.16. *Suppose that Q is a definite map. Then,*

$$Q^{-1}(\partial_W(\text{Conv}(\text{Im } Q))) = \bigcup_{u \in \partial_W K_\psi, u \neq 0} \ker \psi(u) \quad (4.4.3)$$

Proof. This follows from Proposition 4.4.15 and from

$$Q^{-1}(\text{Conv}(Q(\ker \psi(u)))) = \ker \psi(u)$$

\square

The second fact in this section provides a condition under which the boundary $\partial_W \text{Conv}(\text{Im } Q)$ of the convexification of $\text{Im } Q$ is contained in the image $\text{Im } Q$ itself; it is an immediate consequence of Corollary 4.4.11.

Proposition 4.4.17. *Suppose that Q is a definite quadratic map. Then, $\partial_W(\text{Conv}(\text{Im } Q)) \subset \text{Im } Q$ if and only if the following containment holds,*

$$\{p \in W \mid \psi(p^\perp) \text{ is semidefinite}\} \subset \{p \in W \mid \psi(p^\perp) \text{ is nice}\} \quad (4.4.4)$$

The next section will show that round quadratic maps always satisfy the set containment (4.4.4). This has important consequences.

4.5 ROUND QUADRATIC MAPS

This section studies round quadratic maps. It begins with what is by now a familiar formality: a redefinition using the terms of Definition 4.1.1. After asserting some basic properties of round maps, the section begins in earnest by establishing the existence of round maps in $\mathcal{Q}_{n,m}$ for every n and $m \leq n$. After that, the set of round quadratic maps is shown to be an open subset of the set of all quadratic maps; this is an important step towards proving that round maps are *stably* convex. The section concludes by establishing an important correspondence between the boundary $\partial_W(\text{Im } Q)$ of the image of a round quadratic map and the boundary $\partial_W K_\psi$ of the dual of $\text{Im } Q$. Throughout this section, $Q \in \mathcal{Q}(V, W)$ will denote a round quadratic map and $\psi \in \mathcal{L}(W, \text{Sym}(V))$ its ALM.

The following definition generalizes the original definition of roundness given in Chapter 2.

Definition 4.5.1. Q is **round** if it is definite and if $\text{rank } \psi(u) = \dim V - 1$ for all nonzero $u \in \partial_W K_\psi$.

Definition 4.5.2. $\text{Round}(V, W)$ (respectively, $\text{Round}_{n,m}$) is the subset of $\mathcal{Q}(V, W)$ (respectively, $\mathcal{Q}_{n,m}$) consisting of every round quadratic map. The complement of $\text{Round}(V, W)$ in $\mathcal{Q}(V, W)$, the set of quadratic maps that are not round, will be denoted by $\text{NR}(V, W)$ (or, $\text{NR}_{n,m}$ when $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$).

One property of round maps follows immediately from Definition 4.5.1.

Proposition 4.5.3. *If Q is round, then $\ker \psi = \{0\}$.*

Proof. It follows from the definition of K_ψ that $\ker \psi \subset K_\psi$. Thus, $\ker \psi$ must be zero, for any nonzero $u \in \ker \psi$ would be in K_ψ but would have $\text{rank } \psi(u) = 0 < \dim V - 1$. This would contradict the roundness of Q . \square

Corollary 4.5.4. *If Q is round, then $\text{Int}_W(\text{Conv}(\text{Im } Q)) \neq \emptyset$.*

Proof. This follows from Proposition 4.2.5. \square

4.5.1 THE EXISTENCE OF ROUND MAPS

There would be little point in defining or talking about round maps if they did not exist. The next theorem shows that $\text{Round}_n \subset \mathcal{Q}_{n,n}$ is not empty by explicitly constructing a round map in each dimension $n \neq 2$. This suffices to conclude that round maps exist in $\mathcal{Q}_{n,m}$ for $m \leq n$; see Corollary 4.5.7.

Theorem 4.5.5. *Let $n \geq 2$. There exists a round quadratic map in $\mathcal{Q}_{n,n}$.*

The proof of Theorem 4.5.5 requires a basic fact about semidefinite matrices.

Lemma 4.5.6. *A matrix $M \in \text{Sym}_n$ is positive semidefinite with rank k if and only if there exist vectors $v_1, \dots, v_n \in \mathbb{R}^k$ satisfying $\text{span} \{v_1, \dots, v_n\} = \mathbb{R}^k$ and $M_{ij} = \langle v_i, v_j \rangle_{\mathbb{R}^k}$ for $1 \leq i, j \leq n$.*

Proof. See [3] \square

Proof. (Proof of Theorem 4.5.5) If $n = 2$, every nondegenerate definite quadratic map is round.

For $n \geq 3$, consider the quadratic map $Q \in \mathcal{Q}_{n,n}$ whose ALM $\psi \in \mathcal{L}(\mathbb{R}^n, \text{Sym}_n)$ is defined by

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} u_1 & 0 & \dots & \dots & \dots \\ 0 & u_2 & 0 & 0 & \dots \\ 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & u_n \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon u_1 & \varepsilon u_2 & \dots & \varepsilon u_{n-1} \\ \varepsilon u_1 & 0 & \varepsilon u_n & 0 & \dots \\ \varepsilon u_2 & \varepsilon u_n & 0 & 0 & \ddots \\ \vdots & 0 & 0 & \ddots & \ddots \\ \varepsilon u_{n-1} & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

where ε is a nonzero number to be determined. Note that if ε is sufficiently small, the image of the all-ones vector in \mathbb{R}^n under ψ is a positive definite element of Sym_n . Thus, Q is definite for small ε .

Lemma 4.5.6 implies that to prove that Q is round, it suffices to show that for every nonzero $u \in \mathbb{R}^n$, there is no subset $\{v_1, \dots, v_n\} \in \mathbb{R}^{n-2}$ satisfying $\psi(u)_{ij} = \langle v_i, v_j \rangle$ (the subscript \mathbb{R}^{n-2} on the inner products of the v_i will be omitted in the remainder of the proof). If for some nonzero u there was such a subset, then $\psi(u)$ would be positive semidefinite with rank equal to $\dim \text{span} \{v_1, \dots, v_n\} \leq n - 2$, which would contradict the roundness of Q .

Suppose that for some $u \in \mathbb{R}^n$ and some subset $\{v_1, \dots, v_n\} \subset \mathbb{R}^{n-2}$, $\psi(u)_{ij} = \langle v_i, v_j \rangle$. The diagonal elements of $\psi(u)$ then imply that $u_i = \|v_i\|^2$.

Applying the equality $\psi(u)_{ij} = \langle v_i, v_j \rangle$ to the off-diagonal elements of the first row of $\psi(u)$ yields the equalities,

$$\langle v_1, v_i \rangle = \varepsilon u_{i-1} = \varepsilon \|v_{i-1}\|^2 \quad \text{for } i = 2, \dots, n \quad (4.5.1)$$

where $\|v_{i-1}\|^2$ was substituted for u_{i-1} in the final equality.

Applying the equality $\psi(u)_{ij} = \langle v_i, v_j \rangle$ to the (2,3) element of $\psi(u)$ gives,

$$\langle v_2, v_3 \rangle = \varepsilon \|v_n\|^2 \quad (4.5.2)$$

Equations (4.5.1) and (4.5.2) immediately imply two things. First, if $v_1 = 0$, then $v_i = 0$ for $i = 2, \dots, n - 1$. But, if v_2 or v_3 is zero then, by equation (4.5.2), so is v_n . Thus, if $v_1 = 0$, then $v_i = 0$ for all $i = 1, \dots, n$.

Second, if $v_i = 0$ for some $i \in \{2, \dots, n\}$, then equation (4.5.1) implies that $v_j = 0$ for all $j \in \{1, \dots, i\}$. In particular, $v_1 = 0$, from which it follows again that $v_j = 0$ for all $j \in \{1, \dots, n\}$. Thus, if there is a nonzero solution to $\psi(u)_{ij} = \langle v_i, v_j \rangle$, then v_i must be nonzero for all $i \in \{1, \dots, n\}$. It will be assumed from now on that $v_i \neq 0$ for $i \in \{1, \dots, n\}$.

Finally, the remaining elements of $\psi(u)$ imply that

$$\langle v_j, v_k \rangle = 0 \quad \text{for } j, k = 2, \dots, n-1 \text{ with } (j, k) \neq (2, 3) \quad (4.5.3)$$

Equations (4.5.3), together with the fact that $v_i \neq 0$, imply that $\{v_3, \dots, v_n\}$ is an orthogonal basis of \mathbb{R}^{n-2} . Moreover, v_2 is orthogonal to each member of that basis besides v_3 . Thus, $v_2 = \lambda v_3$ for some $\lambda \in \mathbb{R}$.

Equation 4.5.2 now implies that

$$\lambda \|v_3\|^2 = \varepsilon \|v_n\|^2 \quad (4.5.4)$$

Simiarly, the equation $\langle v_1, v_2 \rangle = \varepsilon \|v_1\|^2$ ($i = 2$ in equation 4.5.1) becomes

$$\lambda \langle v_1, v_3 \rangle = \varepsilon \|v_1\|^2 \quad (4.5.5)$$

upon substituing for v_2 . Finally, $\langle v_1, v_3 \rangle = \varepsilon \|v_2\|^2$ ($i = 3$ in equation 4.5.1) implies that

$$\begin{aligned} \langle v_1, v_3 \rangle &= \varepsilon \lambda^2 \|v_3\|^2, \quad \text{or, equivalently,} \\ \lambda \langle v_1, v_3 \rangle &= \varepsilon \lambda^3 \|v_3\|^2 \end{aligned} \quad (4.5.6)$$

Equations (4.5.5) and (4.5.6) imply that $\lambda > 0$. But then Equation 4.5.4 has no solution if $\varepsilon < 0$. Thus, Q is round if $\varepsilon < 0$ (provided that ε is small enough for Q to be definite). \square

Corollary 4.5.7. *Let $n \geq 2$ and $m \leq n$. There exists a round quadratic map in $\mathcal{Q}_{n,m}$.*

Proof. Let n and m be two fixed natural numbers with $m \leq n$. Let $Q_n \in \mathcal{Q}_{n,n}$ denote the round quadratic map that was constructed in the proof of Theorem 4.5.5 and let ψ_n denote its ALM.

The idea underlying this proof is simple. It is this: if Q_n is composed with the orthogonal projection of \mathbb{R}^n onto an m -dimensional subspace that intersects

$\text{Int}_{\mathbb{R}^n} K_{\psi_n}$, the result will be a round quadratic map in $\mathcal{Q}_{n,m}$ (at least, it will be after the m -dimensional subspace is identified with \mathbb{R}^m).

To be precise, let $P \subset \mathbb{R}^n$ be an m -dimensional subspace that intersects $\text{Int}_{\mathbb{R}^n} K_{\psi_n}$, and identify P with \mathbb{R}^m by choosing a basis for it. Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ denote the orthogonal projection onto P , and let $Q = \pi \circ Q_n$.

The fundamental relation (4.1.2) implies that the ALM, ψ_Q , of Q is equal to $\psi_n \circ i_P : \mathbb{R}^m \rightarrow \mathbb{R}^n$, where $i_P : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ is the inclusion map of P . Thus, the image of ψ_Q is equal to $\psi_n(P)$, the image of P under ψ_n . By the choice of P , in particular because $P \cap \text{Int}_{\mathbb{R}^n} K_{\psi_n} \neq \emptyset$, $\psi_n(P)$ contains a positive definite matrix in Sym_n . Thus, Q is definite.

Moreover, Q is necessarily round. This too follows from the equality $\text{Im } \psi_Q = \psi_n(P)$. For, any nonzero semidefinite element in $\text{Im } \psi_Q$ must lie in $\text{Im } \psi_n$ and must therefore have rank $n - 1$. \square

4.5.2 THE STABILITY OF ROUND MAPS

The sole purpose of this section is to prove the following proposition.

Proposition 4.5.8. *Round $_{n,m}$ is open in $\mathcal{Q}_{n,m}$*

Proposition 4.5.8 lightens the burden borne by the proof of the Roundness Theorem. With Proposition 3.6.3 in hand, the proof of the Roundness Theorem will need to demonstrate only that round quadratic maps are convex; their stable convexity will then follow at once from the openness of the set of round maps.

The proof of Proposition 4.5.8 requires a preliminary lemma. It is an immediate consequence of the upper semicontinuity of rank.

Lemma 4.5.9. *Let $S \subset \partial S_n^+$ be a subset consisting only of rank $n - 1$ positive semidefinite matrices. There exists an open set $U \subset \text{Sym}_n$ containing S such that $U \cap \partial S_n^+$ consists only of rank $n - 1$ matrices.*

Suppose that $Q \in \mathcal{Q}_{n,m}$ is a round quadratic map. If $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ is the ALM of Q , then, by definition, $\partial S_n^+ - \{0\} \cap \text{Im } \psi$ consists only of rank $n - 1$ positive semidefinite matrices. Lemma 4.5.9 suggests that ψ can be perturbed

slightly without affecting this property. This is the basic idea underlying the proof of Proposition 4.5.8.

The proof calls on two facts about the compact-open topology that were established long ago, namely Propositions 3.6.3 and 3.6.4. The latter simply says that the compact-open topology on $\mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ is the same as its vector space topology. The proof also invokes the terminology of Section 3.6: $B(K, U)$ denotes the open set of $\mathcal{L}(V, W)$ consisting of those maps mapping the compact set $K \subset V$ into the open set $U \subset W$.

Proof. (Proof of Proposition 4.5.8) Let $\psi \in \text{Sym}(\mathbb{R}^m, \text{Sym}_n)$ be the ALM of a round quadratic map and let $S = \psi(\mathbb{S}^{m-1}) \cap \partial S_n^+$. Because ψ is round, C satisfies the conditions of Lemma 4.5.9. Let U be the open neighbourhood of C provided by the proposition.

Consider the open neighbourhood $B(\mathbb{S}^{m-1}, U \cup (\partial S_n^+)^c) \subset \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ of ψ . Using Proposition 3.6.3, with X taken to be \mathbb{R}^m , Y to be Sym_n , $K = \mathbb{S}^{m-1}$ and $C = \partial S_n^+$, $B(\mathbb{S}^{m-1}, U \cup (\partial S_n^+)^c)$ consists entirely of maps $\psi' \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ for which $\psi'(\mathbb{S}^{m-1}) \cap \partial S_n^+ \subset \psi'(\mathbb{S}^{m-1}) \cap U$. But, U consists entirely of rank $n - 1$ matrices. It follows that every map in the open set $B(\mathbb{S}^{m-1}, U \cup (\partial S_n^+)^c) \cap \text{Def}_n$ (containing ψ) is round. \square

4.5.3 THE BOUNDARY OF $\text{Conv}(\text{Im } Q)$ WHEN Q IS ROUND

When $Q \in \mathcal{Q}(V, W)$ is round, studying the boundary of $\text{Conv}(\text{Im } Q)$ is a fruitful exercise; for example, in the proof of the Roundness Theorem, the convexity of round quadratic maps will be deduced entirely from properties of the boundaries of their images. This section undertakes the exercise, first studying the boundary of $\partial_W K_\psi$ and then establishing an important relationship between it and $\partial_W(\text{Im } Q)$. The upshot of all of this will be that $\partial_W(\text{Im } Q) - \{o\}$ is an embedded submanifold of W . Throughout this section, Q will denote a round map in $\mathcal{Q}(V, W)$, with ALM ψ .

Part of what makes round maps special is that they satisfy the set containment described in Proposition 4.4.17.

Proposition 4.5.10. *Let Q be a round quadratic map. Then,*

$$\partial_W(\text{Conv}(\text{Im } Q)) \subset \text{Im } Q$$

Proof. By Proposition 4.4.17, it suffices to show that

$$\begin{aligned} & \{p \in W \mid \psi(p^\perp) \text{ is semidefinite}\} \\ & \subset \{p \in W \mid \psi(p^\perp) \text{ is nice}\} \end{aligned}$$

The key to this is Proposition 3.5.10, which says that a semidefinite subspace $P \subset \text{Sym}(V)$ that contains a semidefinite element $M \in P$ whose rank is $\dim V - 1$ is necessarily nice. The set containment above follows from this, and from the definition of roundness. \square

Definition 4.5.11. *A quadratic map $Q \in \mathcal{Q}(V, W)$ is **boundary-convex** if $\partial_W(\text{Conv}(\text{Im } Q)) \subset \text{Im } Q$.*

Proposition 4.5.10 says that round quadratic maps are boundary-convex. The only way that a round quadratic map Q could fail to be convex, then, is if there were a p in the (nonempty) *interior* of $\text{Conv}(\text{Im } Q)$ that was not in $\text{Im } Q$. It turns out that this cannot happen, at least when Q maps from \mathbb{R}^n to \mathbb{R}^m with $m \geq 4$ and $n \neq m + 1$. The proof of the Roundness Theorem will show that, in these cases, $\text{Int}_{\mathbb{R}^n}(\text{Conv}(\text{Im } Q)) - \text{Im } Q$ is empty. Loosely speaking, there are no holes in $\text{Im } Q$.

Boundary-convexity is *not* a sufficient condition for convexity. Section 7.1 will describe a (non-round) quadratic map $Q : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ that is boundary-convex, but not convex.

THE BOUNDARY OF $\partial_W K_\psi$

A little more can be said about the boundary of $\text{Conv}(\text{Im } Q)$ when Q is round. When Q is definite, the boundary of $\text{Conv}(\text{Im } Q)$ is the union of the faces $\text{Conv}(Q(\ker \psi(u)))$, with $u \in \partial_W K_\psi - \{0\}$ (see Section 4.4.4). When Q is

round and $u \in \partial_W K_\psi - \{o\}$, $\ker \psi(u)$ is, by the definition of roundness, a 1-dimensional subspace of V . Thus, when Q is round, the boundary of the convexification $\text{Conv}(\text{Im } Q)$ is equal to the union of the half-lines $Q(\ker \psi(u))$, with $u \in \partial_W K_\psi - \{o\}$; all of the faces of $\text{Conv}(\text{Im } Q)$ are 1-dimensional, as convex sets.

To better understand how the half-lines $Q(\ker \psi(u))$, with $u \in \partial_W K_\psi - \{o\}$, are arranged in W , it helps to first look at the boundary of K_ψ . It turns out that $\partial_W K_\psi - \{o\}$ is an embedded submanifold of W . For notational convenience, this is proved for round quadratic maps lying in $\mathcal{Q}_{n,m}$ (that is, mapping from \mathbb{R}^n to \mathbb{R}^m). The proof needs a lemma that makes extensive use of the results in Section 3.8.3, on the smooth structure of the set of rank $n - 1$ matrices in S_n^+ .

Lemma 4.5.12. *Let $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ be the ALM of a round quadratic map $Q \in \mathcal{Q}_{n,m}$. Then, ψ intersects $S_{n,n-1}^+$ transversally.*

Proof. Let $p \in \mathbb{R}^m$ be in the preimage $\psi^{-1}(S_{n,n-1}^+)$ and let $M = \psi(p)$. Because $S_{n,n-1}^+$ has codimension 1, the only way that the equality $T_M \text{Sym}_n = T_M S_{n,n-1}^+ + \psi_{*,p}(T_p \mathbb{R}^m)$ could fail to hold is if $\psi_{*,p}(T_p \mathbb{R}^m)$ were contained in $T_M S_{n,n-1}^+$.

But, ψ is linear, and therefore $\psi_{*,p}(T_p \mathbb{R}^m) = \text{Im } \psi$ (using the usual identification of $T_p \mathbb{R}^m$ with \mathbb{R}^m). So, if ψ is not transverse to $S_{n,n-1}^+$, then $\text{Im } \psi$ is contained in $T_M S_{n,n-1}^+$ (where $T_M S_{n,n-1}^+$ is thought of as a subspace of Sym_n).

But, by Lemma 3.8.10, $T_M S_{n,n-1}^+ = (xx^T)^\perp$, where $x \in \mathbb{R}^n$ is any nonzero element of $\ker M$. By Lemma 4.2.7 and Remark 3.5.5, the containment $\text{Im } \psi \subset (xx^T)^\perp$ implies that $Q(x) = o$. This contradicts the roundness of Q , since round maps are definite and therefore satisfy $Q^{-1}(o) = \{o\}$. \square

Corollary 4.5.13. *Let $Q \in \mathcal{Q}_{n,m}$ be a round quadratic map and let ψ be its ALM. Then, $\partial_{\mathbb{R}^m} K_\psi - \{o\}$ is a submanifold of \mathbb{R}^m . Moreover, if $u \in \partial_{\mathbb{R}^m} K_\psi - \{o\}$, then $T_u(\partial_{\mathbb{R}^m} K_\psi - \{o\}) = \psi^{-1}(T_{\psi(u)} S_{n,n-1}^+) = \psi^{-1}((xx^T)^\perp)$, where $x \in \mathbb{R}^n$ is any nonzero element of the kernel $\ker \psi(u)$.*

Suppose that $H \subset \mathbb{R}^m$ is an affine hyperplane such that $H \cap K_\psi$ is a flat base for K_ψ . It follows from Corollary 4.5.13 that $H \cap K_\psi$ is a convex body in H : it is

certainly convex, it is compact, since K_ψ is closed in \mathbb{R}^m (see Proposition 3.4.18) and it has a nonempty interior, since $\text{Int}_{\mathbb{R}^m} K_\psi \neq \emptyset$. This, together with Proposition 3.8.3, proves the following proposition.

Proposition 4.5.14. *Let $Q \in \mathcal{Q}_{n,m}$ be a round quadratic map and let ψ be its ALM. Then, for any hyperplane $H \subset \mathbb{R}^m$ such that $H \cap K_\psi$ is a flat base for K_ψ , $\partial_H(H \cap K_\psi)$ is diffeomorphic to \mathbb{S}^{m-2} .*

A CERTAIN RELATIONSHIP BETWEEN $\partial_W K_\psi$ AND $\partial_W(\text{Im } Q)$

In general, given a definite quadratic map $Q \in \mathcal{Q}(V, W)$, there is a surjective map from $\partial_W K_\psi - \{o\}$ to the set of faces of $\text{Conv}(\text{Im } Q)$, taking $u \in \partial_W K_\psi - \{o\}$ to the face $u^\perp \cap \text{Conv}(\text{Im } Q) = \text{Conv}(Q(\ker \psi(u)))$. This map can be thought of as being defined on the set of 1-dimensional subspaces spanned by $u \in \partial_W K_\psi - \{o\}$,

$$\{\langle u \rangle : u \in \partial_W K_\psi - \{o\}\} \quad (4.5.7)$$

The remainder of this section is devoted to showing that, when Q is round, the map taking the 1-dimensional subspace $\langle u \rangle$ to the face $Q(\ker \psi(u))$ is *injective*, and therefore a bijection. This fact plays a significant role in the proof of the forward implication (round \Rightarrow stable convexity) in the Roundness Theorem. The bijection between the set in (4.5.7) and the set of faces of $\text{Conv}(\text{Im } Q)$ can be used to infer properties of $\partial_W(\text{Conv}(\text{Im } Q))$ from properties of $\partial_W K_\psi$. This turns out to be advantageous; $\partial_W K_\psi$ is a little easier to study directly than $\partial_W(\text{Conv}(\text{Im } Q))$.

Remark 4.5.15. The natural setting for the preceding arguments, and of some those to come, is the projective space of W . This point of view is not taken here for the sake of simplicity. It is taken, however, in Section 5.4, where it will be shown that the map $\langle u \rangle \mapsto Q(\ker \psi(u))$ corresponds to a diffeomorphism of submanifolds of \mathbb{RP}^{n-1} . ■

For the purposes of this section, and this section only, let ΔK_ψ denote the

collection of 1-dimensional subspaces in (4.5.7) and let $\mathcal{F}(\text{Conv}(\text{Im } Q))$ denote the set of faces of $\text{Conv}(\text{Im } Q)$. The main assertion in this section can now be stated as this,

Proposition 4.5.16. *The map from ΔK_ψ to $\mathcal{F}(\text{Conv}(\text{Im } Q))$ taking $\langle u \rangle$ to the face $Q(\ker \psi(u))$ is a bijection.*

The map in Proposition 4.5.16 is surjective by virtue of Q being definite. It can fail to be injective in only two ways: either there exist distinct $\langle u_1 \rangle$ and $\langle u_2 \rangle$ with $\ker \psi(u_1) = \ker \psi(u_2)$, or there exist distinct kernels $\ker \psi(u_1)$ and $\ker \psi(u_2)$ which are such that $Q(\ker \psi(u_1)) = Q(\ker \psi(u_2))$. The remainder of this section is devoted to proving that neither of these alternatives is possible.

Proposition 4.5.17. *Suppose that Q is a round quadratic map. If $u_1, u_2 \in \partial_W K_\psi$ are linearly independent, then $\ker \psi(u_1) \neq \ker \psi(u_2)$.*

Proposition 4.5.18. *Suppose that Q is a round quadratic map. If $u_1, u_2 \in \partial_W K_\psi$ are nonzero and if $Q(\ker \psi(u_1)) = Q(\ker \psi(u_2))$, then $\langle u_1 \rangle = \langle u_2 \rangle$ and therefore $\ker \psi(u_1) = \ker \psi(u_2)$.*

These are proved in succession.

PROOF OF THE FIRST PROPOSITION, PROPOSITION 4.5.17

What follows is a proof of a statement that is more general than the one in Proposition 4.5.17; it allows for the possibility that $\psi(u_2)$ might be indefinite.

Proposition 4.5.19. *Suppose that $Q \in \mathcal{Q}(V, W)$ is a round quadratic map with $ALM \psi$. If $u_1 \in \partial_W K_\psi$ and $u_2 \in W$ are linearly independent, then $\ker \psi(u_1)$ is not contained in $\ker \psi(u_2)$.*

Proposition 4.5.19 implies Proposition 4.5.17. It will be summoned later on, in the proof of the Roundness Theorem.

Proposition 4.5.19 is proved with the help of three lemmas. The first says that the boundary of a proper (proper as a subset), closed cone always contains more than the origin.

Lemma 4.5.20. *Let V be a finite-dimensional real inner product space with $\dim V > 1$. Let $K \subset V$ be a closed convex cone with $\text{Int}_V(K) \neq \emptyset$. If $K \neq V$, then $\partial_V K \neq \{0\}$.*

Proof. When $\dim V > 1$, $V - \{0\}$ is connected. Because K is convex, $K = \text{Cl}_V(\text{Int}_V(K))$. If $\partial_V K = \{0\}$, it would follow that $K - \{0\} = \text{Int}_V(K)$. $K - \{0\}$, therefore, would be a closed and open subset of $V - \{0\}$. The connectedness of $V - \{0\}$ would then imply that $K - \{0\} = V - \{0\}$ and therefore that $K = V$. □

The next lemma says that the linear span of a positive definite matrix and an indefinite matrix necessarily contains a strictly semidefinite matrix.

Lemma 4.5.21. *Let V be a finite-dimensional real inner product space. If $M_1 \in \mathring{S}^+(V)$ and $M_2 \in \text{Sym}(V)$ are linearly independent, then $\text{span}\{M_1, M_2\}$ intersects $\partial S^+(V)$ nontrivially.*

Proof. Let $P = \text{span}\{M_1, M_2\}$. Then, $K = P \cap S^+(V)$ is a closed cone with $\text{Int}_P(K) \neq \emptyset$ (since M_1 is in $\text{Int}_P(K)$). Furthermore, $K \neq P$. Lemma 4.5.20 implies that the boundary of K is nonzero. The boundary must contain a strictly semidefinite element of $\text{Sym}(V)$, since $P \cap \mathring{S}^+(V)$ consists entirely of interior points of K (relative to P). □

Lemma 4.5.22. *Let $M_1 \in S^+(V)$ and $M_2 \in \text{Sym}(V)$ be linearly independent points with $\text{rank } M_1 = \dim V - 1$. If $\ker M_1 \subset \ker M_2$, there exists an $M \in \text{span}\{M_1, M_2\} \cap S^+(V)$ such that $0 < \text{rank } M < \dim V - 1$.*

Proof. Let $l = \ker M_1 \subset \ker M_2$ and let \overline{M}_1 and \overline{M}_2 denote the restrictions of M_1 and M_2 to l^\perp (l^\perp is invariant under \overline{M}_1 and \overline{M}_2). Then, $\overline{M}_1 \in \mathring{S}^+(l^\perp)$ and $\overline{M}_2 \in \text{Sym}(l^\perp)$, and \overline{M}_1 and \overline{M}_2 are linearly independent. By Proposition 4.5.21, $\text{span}\{\overline{M}_1, \overline{M}_2\}$ intersects $\partial_{\text{Sym}(l^\perp)} S^+(l^\perp)$ nontrivially, say at $\overline{M} = \alpha \overline{M}_1 + \beta \overline{M}_2 \in \text{Sym}(l^\perp)$. Note that $0 < \text{rank } \overline{M} \leq \dim V - 2$. Let

$M \in \text{Sym}(V)$ be defined by

$$M = \begin{cases} 0 & \text{on } l \\ i_{l^\perp} \circ \overline{M} & \text{on } l^\perp \end{cases}$$

where i_{l^\perp} is the inclusion of l^\perp into V . Then, $M = \alpha M_1 + \beta M_2$ and $0 < \text{rank } M < \dim V - 1$. □

Proposition 4.5.19 now follows, for if two linearly independent elements $u_1 \in \partial_W K_\psi$ and $u_2 \notin K_\psi$ were such that $\ker \psi(u_1) \subset \ker \psi(u_2)$, then $0 < \text{rank } \psi(u) < \dim V - 1$ for some $u \in \text{span} \{u_1, u_2\} \cap K_\psi$. This contradicts the roundness of Q .

PROOF OF THE SECOND PROPOSITION, PROPOSITION 4.5.18

The proof of Proposition 4.5.18 invokes the following fact.

Lemma 4.5.23. *Let M_1, M_2 be linearly independent points in $S^+(V)$ with $\text{rank } M_1 = \text{rank } M_2 = \dim V - 1$. If $\text{span} \{M_1, M_2\} \cap S^+(V)$ is contained in ∂S_n^+ , then $\ker M_1 = \ker M_2$.*

Proof. If $\text{span} \{M_1, M_2\} \cap S^+(V)$ is contained in ∂S_n^+ , then the cone $\text{span} \{M_1, M_2\} \cap S^+(V)$ is a subset of a face $F = S^+(V) \cap H$ where $H \subset \text{Sym}(V)$ is a hyperplane (see Remark 3.5.8). But, each M_i , having rank $\dim V - 1$, is contained in only one face of $S^+(V)$, namely $S^+(V) \cap (x_i \otimes x_i)^\perp$ where x_i is a nonzero element of $\ker M_i$ (recall that, for $x \in V$, $x \otimes x$ denotes the rank 1 symmetric map in $\text{Sym}(V)$ corresponding to x). Since M_1 and M_2 are both in F , it must be the case that $H = (x_1 \otimes x_1)^\perp = (x_2 \otimes x_2)^\perp$. The last equality implies that $\ker M_1 = \ker M_2$. □

Suppose now that $Q(\ker \psi(u_1)) = Q(\ker \psi(u_2))$ for some linearly independent $u_1, u_2 \in \partial_W K_\psi$ and let $y \in W$ be any nonzero element of $Q(\ker \psi(u_1))$. Note that by virtue of $y \in \text{Im } Q$, $\langle y, u \rangle_W \geq 0$ for all $u \in K_\psi$, with equality if and only if $u \in \partial_W K_\psi$.

Because $y \in Q(\ker \psi(u_1))$ and $y \in Q(\ker \psi(u_2))$, the fundamental relation (4.1.2) implies that $\langle y, u_1 \rangle_V = \langle y, u_2 \rangle_V = 0$. It follows that the linear span of u_1 and u_2 is orthogonal to y , and therefore that $\text{span} \{u_1, u_2\} \cap K_\psi \subset \partial_W K_\psi$. But this is equivalent to $\text{span} \{\psi(u_1), \psi(u_2)\} \subset \partial S^+(V)$. By Lemma 4.5.23, $\ker \psi(u_1)$ and $\ker \psi(u_2)$ must therefore be equal. As it did in the proof of Proposition 4.5.17, the equality $\ker \psi(u_1) = \ker \psi(u_2)$ contradicts the roundness of Q (again because of Lemma 4.5.22). This completes the proof of Proposition 4.5.18.

4.6 EQUIVALENCE OF QUADRATIC MAPS

Quadratic maps can be manipulated in ways that make them easier to study. An important example of this is the pre- and post-composition of quadratic maps with linear isomorphisms: if $Q : V \rightarrow W$ is a quadratic map, and $\phi : V \rightarrow V$, $\tau : W \rightarrow W$ linear isomorphisms, then $Q' = \tau \circ Q \circ \phi$ is also a quadratic map. Moreover, Q is convex if and only if Q' is convex. Thus, for the purposes of any inquiry into the convexity of Q or Q' , the two are interchangeable.

Definition 4.6.1. Quadratic maps $Q, Q' \in \mathcal{Q}(V, W)$ are **equivalent** if there exist linear isomorphisms $\phi : V \rightarrow V$ and $\tau : W \rightarrow W$ such that $Q' = \tau \circ Q \circ \phi$.

Proposition 4.6.2. Let $Q, Q' \in \mathcal{Q}(V, W)$ be quadratic maps and let $\psi, \psi' \in \mathcal{L}(W, \text{Sym}(V))$ be their associated linear maps. If Q and Q' are equivalent, and related by $Q' = \tau \circ Q \circ \phi$ for some linear isomorphisms $\tau : W \rightarrow W$ and $\phi : V \rightarrow V$, then ψ and ψ' are related by $\psi'(u) = \phi \cdot (\psi \circ \tau)(u) \cdot \phi^*$ where \cdot denotes composition in $\mathcal{L}(V, V)$ and ϕ^* denotes the adjoint of ϕ .

In particular, if $\phi = I_V$, then $\text{Im } \psi = \text{Im } \psi'$.

Proof. This follows from the fundamental relation (4.1.2). □

Remark 4.6.3. Note that Proposition 4.6.2 implies that, in addition to convexity, both definiteness and roundness are preserved by the equivalence relation in 4.6.1 ■

Remark 4.6.4. Let $Q, Q' : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be quadratic maps that are defined respectively by $M_1, \dots, M_m \in \text{Sym}_n$ and $M'_1, \dots, M'_m \in \text{Sym}_n$. The second part of Proposition 4.6.2 implies that Q and Q' are related by $Q' = \tau \circ Q$ for some isomorphism $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ if and only if $\text{span} \{M_1, \dots, M_m\} = \text{span} \{M'_1, \dots, M'_m\}$. ■

Suppose now that $Q \in \mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ is a definite quadratic map defined by $M_1, \dots, M_m \in \text{Sym}_n$ and $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ its ALM. For some $u \in \mathbb{R}^m$, $\psi(u)$ is positive definite. It follows that there exists a linear isomorphism φ of \mathbb{R}^n which is such that $\varphi \cdot \psi(u) \cdot \varphi^* = I_n$. Thus, there is a basis $\{M'_1, \dots, M'_m\}$ of the subspace $\varphi \cdot \text{Im } \psi \cdot \varphi^* := \text{span} \{\varphi \cdot M_1 \cdot \varphi^*, \dots, \varphi \cdot M_m \cdot \varphi^*\}$ in which $M'_m = I_n$. It follows from Proposition 4.6.2 that every definite quadratic map in $\mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$ is equivalent to a map of the form

$$x \in \mathbb{R}^n \mapsto \begin{pmatrix} \langle x, M'_1 x \rangle \\ \vdots \\ \langle x, M'_{n-1} x \rangle \\ \|x\|^2 \end{pmatrix} \quad (4.6.1)$$

whose last coordinate is just the squared norm of $x \in \mathbb{R}^n$.

It follows that it suffices to prove the Roundness Theorem for quadratic maps that have the form 4.6.1. This is what is done in the next chapter.

INDEPENDENCE FROM THE INNER PRODUCTS \langle, \rangle_V AND \langle, \rangle_W

This section concludes by affirming that Definitions 4.4.1 and 4.5.1, those of definiteness and roundness respectively, are independent of the inner products \langle, \rangle_V and \langle, \rangle_W . Suppose then that \langle, \rangle'_V and \langle, \rangle'_W are inner products on V and W , possibly distinct from \langle, \rangle_V and \langle, \rangle_W . Let $Q \in \mathcal{Q}(V, W)$ be a quadratic map and

let $\psi', \psi \in \mathcal{L}(W, \text{Sym}(V))$ be the ALMs defined by each pair of inner products,

$$\begin{aligned}\langle Q(x), u \rangle_W &= \langle x, \psi(u)x \rangle_V \\ \langle Q(x), y \rangle'_W &= \langle x, \psi'(u)x \rangle'_V\end{aligned}$$

Let $G_V \in \mathcal{L}(V, V)$ and $G_W \in \mathcal{L}(W, W)$ denote the linear isomorphisms satisfying

$$\begin{aligned}\langle x, y \rangle'_V &= \langle x, G_V y \rangle_V \quad \text{and} \\ \langle u, v \rangle'_W &= \langle u, G_W v \rangle_W\end{aligned} \tag{4.6.2}$$

for all $x, y \in V$ and $u, v \in W$.

Remark 4.6.5. Equation (4.6.2) implies that a linear map $\tau \in \mathcal{L}(V, V)$ is positive semidefinite with respect to $\langle \cdot, \cdot \rangle'_V$ if and only if $G_V \cdot \tau$ is positive semidefinite with respect to $\langle \cdot, \cdot \rangle$, where \cdot denotes multiplication in $\mathcal{L}(V, V)$. ■

If $x \in V$ and $u \in W$, then,

$$\begin{aligned}\langle Q(x), u \rangle'_W &:= \langle x, \psi'(u)x \rangle'_V \\ &= \langle x, G_V \psi'(u)x \rangle_V\end{aligned} \tag{4.6.3}$$

where the first equality is by the definition of ψ' and the second by (4.6.2). On the other hand, using (4.6.2) again,

$$\begin{aligned}\langle Q(x), u \rangle'_W &= \langle Q(x), G_W u \rangle_W \\ &= \langle x, \psi(G_W u)x \rangle_V\end{aligned} \tag{4.6.4}$$

where the second equality follows from the definition of ψ . Equations (4.6.3) and (4.6.4) provide the equality $G_V \cdot \psi' = \psi \circ G_W$.

Three facts follow from the equality $G_V \cdot \psi' = \psi \circ G_W$: first, by Remark 4.6.5, $\psi'(u)$ is positive semidefinite with respect to $\langle \cdot, \cdot \rangle'_V$ if and only if $\psi(u)$ is positive semidefinite with respect to $\langle \cdot, \cdot \rangle_V$, secondly that $\psi'(u)$ and $\psi(u)$ have the same

rank for all $u \in W$, and lastly that $\ker \psi' = \{0\}$ if and only if $\ker \psi = \{0\}$. These facts prove in succession that the definitions of definiteness, roundness and nondegeneracy are independent of the choice of inner product on V and W .

4.7 APPENDIX: A PROOF OF PROPOSITION 4.3.2

This section presents a proof of Proposition 4.3.2, which is reproduced below for convenience. None of the ideas that the proof invokes are used elsewhere in this thesis.

Proposition 4.3.2. *The compact-open topology on $\mathcal{Q}(V, W)$ coincides with the finite-dimensional vector space topology induced by the bijection $\rho : \mathcal{Q}(V, W) \rightarrow \mathcal{L}(W, \text{Sym}(V))$, which associates each quadratic map $Q \in \mathcal{Q}(V, W)$ with its ALM $\psi \in \mathcal{L}(W, \text{Sym}(V))$.*

Let τ_{co} denote the compact-open topology on $\mathcal{Q}(V, W)$, and let τ_{vs} denote the finite-dimensional real vector space topology on $\mathcal{L}(W, \text{Sym}(V))$. Because it is a set of continuous maps, $\mathcal{L}(W, \text{Sym}(V))$ also carries a compact-open topology. By Proposition 3.6.4, the compact-open topology on $\mathcal{L}(W, \text{Sym}(V))$ coincides with τ_{vs} . In terms of these topologies, Proposition 4.3.2 can be cast as follows,

Proposition 4.7.1. *With $\mathcal{Q}(V, W)$ and $\mathcal{L}(W, \text{Sym}(V))$ topologized as just described, the bijection ρ is a homeomorphism.*

The forthcoming proof will be done in the terms of Proposition 4.7.1, rather than those of the original Proposition, 4.3.2. The proof of Lemma 4.7.1 proceeds by producing two subbases, one for τ_{co} and one for τ_{vs} . Each member of S_{co} , the subbase for τ_{co} , is mapped by ρ to an open set in $\mathcal{L}(W, \text{Sym}(V))$. Similarly, each member of S_{vs} , the subbase for τ_{vs} , is mapped to an open set in $\mathcal{Q}(V, W)$ by ρ^{-1} . The existence of S_{co} and S_{vs} suffice to prove the Proposition.

The definitions of S_{co} and S_{vs} require a few preliminary observations. Recall Proposition 3.6.2,

Proposition 3.6.2. *Let S be a subbase for the topology on Y and let $Q \subset \mathcal{C}(X, Y)$. Then, the collection $\{B(K, U) \cap Q \mid K \subset X \text{ compact, } U \in S\}$ also generates the*

compact-open topology on Q .

The next proposition provides subbases that can be used in conjunction with Proposition 3.6.2. It requires some notation,

Definition 4.7.2. Let L be a finite-dimensional inner product space. Then, $H_u^{\beta,-}$ is the open half-space of L given by

$$H_u^{\beta,-} = \{l \in L \mid \langle l, u \rangle_L < \beta\}$$

Proposition 4.7.3. Let L be a finite-dimensional real inner product space and let $B = \{l_i\}_{i=1}^{\dim L}$ be a basis of L . Then, the collection $\{H_{l_i}^{\beta,-} \mid l_i \in B, \beta \in \mathbb{R}\}$ is a subbasis for the vector space topology on L .

Proof. The collection of semi-infinite subintervals $(-\infty, a)$ and (b, ∞) with $a, b \in \mathbb{R}$ is a subbasis for \mathbb{R} . It follows that when $L = \mathbb{R}^n$ and when B is the standard basis, the collection of subsets described in the lemma is a subbasis. The result now holds for any L . \square

Returning to the proof of Lemma 4.7.1, let $B_W = \{w_j\}$ be a basis for W and let $B_{\text{Sym}(V)} = \{q_{v_i}\}$ be a basis of $\text{Sym}(V)$ consisting entirely of rank 1 maps (such bases do exist; for example, $\{(e_i + e_j)(e_i + e_j)^T\}$ is a basis of Sym_n , where $\{e_i\}$ is the standard basis of \mathbb{R}^n).

Propositions 3.6.2 and 4.7.3 reveal that

$$S_{co} = \left\{ B(K, H_{w_j}^{\beta,-}) \mid \beta \in \mathbb{R}, w_j \in B_W, K \subset V \text{ compact} \right\} \quad \text{and} \\ S_{vs} = \left\{ B(K, H_{q_{v_i}}^{\beta,-}) \mid \beta \in \mathbb{R}, q_{v_i} \in B_{\text{Sym}(V)}, K \subset W \text{ compact} \right\}$$

are subbases for the compact-open topologies on $\mathcal{Q}(V, W)$ and $\mathcal{L}(W, \text{Sym}(V))$. These are the sought after subbases that were alluded to earlier. Showing that every member of each subbase maps to an open set in the others topology (under ρ and ρ^{-1}) requires one last definition. Definition 4.7.4 generalizes the notation $H_u^{\beta,-}$.

Definition 4.7.4. Let L be a finite-dimensional, real vector space and let $C \subset L$ be a compact subset. Then, $H_C^{\beta, -}$ is the subset

$$H_C^{\beta, -} = \{l \in L \mid \langle l, c \rangle < \beta \forall c \in C\}$$

Proposition 4.7.5. For any compact $C \subset L$, $H_C^{\beta, -}$ is open in L .

Proof. This follows from the compactness of C and the continuity of the map from $L \times C$ to \mathbb{R} given $(l, c) \mapsto \langle l, c \rangle$. \square

Proof. (Proof of Proposition 4.7.1) Let $B(K, H_{w_j}^{\beta, -})$ be an element of S_{co} . A quadratic map Q is in $B(K, H_{w_j}^{\beta, -})$ if and only if $\langle Q(x), w_j \rangle_W < \beta$ for all $x \in K \subset V$. The fundamental relation (4.1.2) then implies that $\psi \in \mathcal{L}(W, \text{Sym}(V))$ is in $\rho(B(K, H_{w_j}^{\beta, -}))$ if and only if $\langle x, \psi(w_j)x \rangle_V < \beta$ for all $x \in K$. But, $\langle x, \psi(w_j)x \rangle_V = \langle x \otimes x, \psi(w_j) \rangle_{\text{Sym}(V)}$, and so, $\langle x \otimes x, \psi(w_j) \rangle_{\text{Sym}(V)} < \beta$.

But, $\langle x \otimes x, \psi(w_j) \rangle_{\text{Sym}(V)} < \beta$ is a necessary and sufficient condition for $\psi(w_j) \in H_{q(K)}^{\beta, -}$ (using the notation of Definition 4.7.4, and the map $q : V \rightarrow \text{Sym}(V)$ defined in Section 3.2). Thus, $\rho(B(K, H_{w_j}^{\beta, -})) = B(\{w_j\}, H_{q(K)}^{\beta, -})$ which, by Proposition 3.6.4 is open in $\mathcal{L}(W, \text{Sym}(V))$.

Conversely, if $B(K, H_{q_{v_i}}^{\beta, -})$ is an element of S_{vs} , and if $\psi \in B(K, H_{q_{v_i}}^{\beta, -})$, then $\langle v_i \otimes v_i, \psi(x) \rangle_{\text{Sym}(V)} < \beta$ for all $x \in K \subset W$. But, $\langle v_i \otimes v_i, \psi(x) \rangle_{\text{Sym}(V)} = \langle v_i, \psi(x)v_i \rangle_V$. So, $\langle v_i, \psi(x)v_i \rangle_V < \beta$. Then, again by the fundamental relation (4.1.2), $\langle Q(v_i), x \rangle < \beta$ for all $x \in K$ where Q is the quadratic map associated with ψ . But, this is a necessary and sufficient condition for Q to be in $B(\{v_i\}, H_K^{\beta, -})$. Thus, $\rho^{-1}(B(K, H_{q_{v_i}}^{\beta, -})) = B(\{v_i\}, H_K^{\beta, -})$, which is open in $\mathcal{Q}(V, W)$. This completes the proof. \square

5

A Proof of the Roundness Theorem

This chapter proves the Instrumental Lemma, and the Roundness Theorem, in that order. Each proof is accompanied by an example that demonstrates the main ideas underlying the proof. The chapter begins by restating the Instrumental Lemma and the Roundness Theorem in the terms of the previous chapter.

5.1 THE ROUNDNESS THEOREM AND THE INSTRUMENTAL LEMMA, REVISITED

Both the Roundness Theorem and the Fundamental Lemma can be restated in ways that make them slightly easier to work with; both are restated here. The proofs that appear later in this chapter will occasionally refer to the Roundness Theorem and the Instrumental Lemma in the forms that they take below.

In the terms of Chapter 4, the Roundness Theorem says that when $m \leq n$,

$m \neq n - 1$, the interior of the set of convex, definite quadratic maps in $\mathcal{Q}_{n,m}$ is exactly the set $\text{Round}_{n,m}$ of round quadratic maps.

Theorem 5.1.1. (*The Roundness Theorem, Version 2*) *Let n, m be natural numbers with $4 \leq m \leq n$, $m \neq n - 1$. Then, $\text{Int}_{\mathcal{Q}_{n,m}}(\text{Convex}_{n,m} \cap \text{Def}_{n,m}) = \text{Round}_{n,m}$.*

Meanwhile the Instrumental Lemma says that when $n < m$, the interior of the set of convex, definite quadratic maps is empty.

Lemma 5.1.2. (*The Instrumental Lemma, Version 2*) *Let n, m be natural numbers with $2 \leq n < m$. Then, $\text{Int}_{\mathcal{Q}_{n,m}}(\text{Convex}_{n,m} \cap \text{Def}_{n,m}) = \emptyset$.*

5.2 A PROOF OF THE INSTRUMENTAL LEMMA

The Instrumental Lemma says that, when $2 \leq n < m$, the interior of $\text{Convex}_{n,m} \cap \text{Def}_{n,m}$ is empty. Thus, to prove the Instrumental Lemma, it suffices to find a subset of the complement $(\text{Convex}_{n,m} \cap \text{Def}_{n,m})^c \subset \mathcal{Q}_{n,m}$ that is dense in $\mathcal{Q}_{n,m}$. That is, it suffices to find a dense subset of $\mathcal{Q}_{n,m}$, none of whose members is convex. The following definition hints at such a subset.

Definition 5.2.1. *A quadratic map $Q \in \mathcal{Q}(V, W)$ is **nondegenerate** if its associated linear map $\psi \in \mathcal{L}(W, \text{Sym}(V))$ has full rank.*

Example 5.2.2. *If Q is a quadratic map from \mathbb{R}^n to \mathbb{R}^m defined by the symmetric matrices $M_1, \dots, M_m \in \text{Sym}_n$, nondegeneracy means that $\dim \text{span} \{M_1, \dots, M_m\}$ is as large as possible. ■*

If the set of nondegenerate quadratic maps in $\mathcal{Q}_{n,m}$ is going to be the vehicle for the proof of the Instrumental Lemma, it must have two properties: it must be dense, and it must not contain a single convex quadratic map. The next proposition affirms the first property.

Proposition 5.2.3. *The set of nondegenerate quadratic maps in $\mathcal{Q}(V, W)$ is open and dense.*

Proof. The set of full-rank maps in $\mathcal{L}(W, \text{Sym}(V))$ is open and dense. The corollary now follows from Proposition 4.3.2, which says that $\mathcal{L}(W, \text{Sym}(V))$, with its vector space topology, is homeomorphic to $\mathcal{Q}(V, W)$, with its compact-open topology. \square

The remainder of this section is devoted to showing that no nondegenerate quadratic map in $\mathcal{Q}_{n,m}$ is convex when $n < m$. This will complete the proof of the Instrumental Lemma. From now on, Q will denote a nondegenerate quadratic map in $\mathcal{Q}_{n,m}$ with $2 \leq n < m$, and $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ will denote its associated linear map.

If Q is nondegenerate, then ψ is either injective, or surjective but not injective. These cases will be treated separately, in that order.

WHEN ψ IS INJECTIVE

When ψ is injective, the proof that Q is not convex is straightforward. Moreover, it contains all of the essential ideas underlying the Instrumental Lemma, and it will serve as the basis for the proof that Q is not convex when ψ is not injective.

When ψ is injective, Q fails to be convex for a simple reason: on one hand, the interior $\text{Int}_{\mathbb{R}^m}(\text{Im } Q)$ of the image of Q is empty, while on the other hand, the interior $\text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$ of the convexification of $\text{Im } Q$ is *not* empty. Thus, when ψ is injective, $\text{Im } Q \neq \text{Conv}(\text{Im } Q)$, and Q is not convex. Of course, these assertions need to be verified.

Given that $\ker \psi = \{0\}$, the nonemptiness $\text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q)) \neq \emptyset$ is a familiar fact; it follows from Proposition 4.2.5.

The emptiness $\text{Int}_{\mathbb{R}^m}(\text{Im } Q) = \emptyset$ is a consequence of Sard's Theorem, in particular, of the special version of Sard's Theorem that appeared as Theorem 3.8.23. This proves the Instrumental Lemma when ψ is injective.

WHEN ψ IS SURJECTIVE, BUT NOT INJECTIVE

When ψ is surjective but not injective, it is possible to find a surrogate nondegenerate quadratic map \bar{Q} whose image is the same as that of Q , but whose

associated linear map is injective. The map \overline{Q} is then subject to the results of the previous section. It is not convex, and neither, then, is Q (since $\text{Im } \overline{Q} = \text{Im } Q$). The remainder of this section supplies the details of this argument.

The image of Q is contained in the subspace $(\ker \psi)^\perp$; this is by Proposition 4.2.2 and is an immediate consequence of the fundamental relation (4.1.2). Since ψ is a linear map, $\dim(\ker \psi)^\perp = \dim \text{Im } \psi$. When ψ is surjective, $\dim \text{Im } \psi = \dim \text{Sym}(V)$. When $\dim V \geq 2$, $\dim \text{Sym}(V)$ is strictly greater than $\dim V$. Thus, $\dim(\ker \psi)^\perp > \dim V$.

Let $\pi : W \rightarrow (\ker \psi)^\perp$ denote the orthogonal projection of W onto $(\ker \psi)^\perp$. From what was just observed, the quadratic map $\overline{Q} = \pi \circ Q : V \rightarrow (\ker \psi)^\perp$ maps from V to a space whose dimension is strictly greater than that of V . Moreover, the image of Q is equal to the image of $i \circ \overline{Q}$, where $i : (\ker \psi)^\perp \rightarrow W$ is the inclusion map. Thus, Q is convex if and only if \overline{Q} is convex. It now suffices to show that \overline{Q} is not convex, and for this, it suffices to show that the ALM $\overline{\psi}$ of \overline{Q} is injective. This will occupy the rest of this section and will complete the proof of the Instrumental Lemma.

The following proposition (Proposition 5.2.5) describes the relationship between the ALM $\overline{\psi}$ of \overline{Q} and that, ψ , of Q . The proposition will imply that $\overline{\psi}$ is injective (see Remark 5.2.7 below). The setting of the proposition is slightly more general than is needed for present purposes; this is done for future use.

Remark 5.2.4. The upcoming proposition, Proposition 5.2.5, requires some new notation. Let V be an inner product space and $L \subset V$ a linear subspace. There is a map from $\text{Sym}(V)$ to $\text{Sym}(L)$ which takes an $M \in \text{Sym}(V)$ and sends it to $\pi_L \circ M \circ i_L \in \text{Sym}(L)$ where $i_L : L \hookrightarrow V$ is the inclusion map and $\pi_L : V \rightarrow L$ the orthogonal projection. This map will be denoted by R_L (the R stands for restriction).

If $V = \mathbb{R}^n$ and $L = \mathbb{R}^k \subset V$ is the subspace of points whose last $n - k$ coordinates vanish, then, for $M \in \text{Sym}_n$, $R_L(M)$ is the upper $k \times k$ block of M . ■

Proposition 5.2.5. *Let $Q \in \mathcal{Q}(V, W)$ be a quadratic map with ALM $\psi \in \mathcal{L}(W, \text{Sym}(V))$. Let $L \subset V$ be a linear subspace and suppose that $Q(L)$ is*

contained in a subspace $H \subset W$. Let $\bar{Q} = \pi_H \circ Q \circ i_L : L \rightarrow H$ denote the composition of Q with the inclusion map $i : L \rightarrow V$, followed by the orthogonal projection π_H . Let $\bar{\psi} : H \rightarrow \text{Sym}(L)$ denote the ALM of \bar{Q} . Then, $\bar{\psi} = R_L \circ \psi \circ i_H$ where $i_H : H \hookrightarrow W$ is the inclusion and $R_L : \text{Sym}(V) \rightarrow \text{Sym}(L)$ is the restriction map defined in Remark 5.2.4.

Remark 5.2.6. Note that for all $z \in W$ and $u \in H$, $\langle \pi_H z, u \rangle_H = \langle z, i_H u \rangle_W$. That is, i_H and π_H are adjoint. This fact will be used in the following proof. ■

Proof. For the first part of the lemma, let $x \in L$ and $u \in H$. Then,

$$\begin{aligned} \langle \bar{Q}(x), u \rangle_H &= \langle (\pi_H \circ Q \circ i_L)(x), u \rangle_H \\ &= \langle Q(i_L(x)), i_H(u) \rangle_W \\ &= \langle i_L(x), (\psi \circ i_H)(u) i_L(x) \rangle_V \\ &= \langle x, (\pi_L \circ (\psi(i_H(u))) \circ i_L)x \rangle_V \\ &= \langle x, (R_L \circ \psi \circ i_H)(u)x \rangle_L \end{aligned}$$

The second equality is by Remark 5.2.6, the third by the definition of the ALM of Q , the fourth again by Remark 5.2.6 and the fifth by the definition of the map R_L . The final equality, $\langle \bar{Q}(x), u \rangle_H = \langle x, (R_L \circ \psi \circ i_H)(u)x \rangle_L$ is precisely the one satisfied by the ALM of \bar{Q} . Thus, $\bar{\psi} = R_L \circ \psi \circ i_H$. □

Remark 5.2.7. If $L = V$ in Proposition 5.2.5, then $\bar{\psi} = \psi \circ i_H$. Thus, the ALM of $\pi_H \circ Q$ is just the restriction of the ALM of Q to H . In particular, if $H = (\ker \psi)^\perp$, then $\bar{\psi}$ is injective. This proves the Instrumental Lemma in the case that ψ is surjective but not injective.

Even if $L \neq V$, the reasoning of the previous paragraph can be applied directly to $Q|_L$ to reach this conclusion: the ALM of $\pi_H \circ Q|_L$ is the restriction of the ALM of $Q|_L$ to H . This fact will be used in the next section. ■

Let $L \subset V$ and $H \subset W$ be subspaces as in Proposition 5.2.5. The map $\Gamma_{LH} : \text{Sym}(W, \text{Sym}(V)) \rightarrow \text{Sym}(H, \text{Sym}(L))$ taking the ALM of Q to the ALM

of $\bar{Q} = \pi_H \circ Q \circ i_L$ has some basic properties that will prove useful later on in this chapter.

Proposition 5.2.8. *The map $\Gamma_{LH} : \mathcal{L}(W, \text{Sym}(V)) \rightarrow \mathcal{L}(H, \text{Sym}(L))$ taking ψ to $\bar{\psi} = R_L \circ \psi \circ i_H$ is a linear surjection and therefore an open map (i.e. Γ_{LH} maps open sets to open sets). The kernel of Γ_{LH} contains all linear maps $\psi \in \mathcal{L}(W, \text{Sym}(V))$ that vanish on H .*

Proof. The map Γ_{LH} is given by $\psi \mapsto \bar{\psi} = R_L \circ \psi \circ i_H$, where R_L is the map defined in Remark 5.2.4. Γ_{LH} is linear. To show that it is surjective, let τ be an element of $\mathcal{L}(H, \text{Sym}(L))$. Define $\varphi \in \mathcal{L}(W, \text{Sym}(V))$ as follows :

$$\varphi(h) = \begin{cases} 0 & \text{if } h \in H^\perp \\ i_L \circ \tau(h) \circ \pi_L & \text{if } h \in H \end{cases}$$

where $i_L : L \rightarrow V$ is the inclusion map of L and $\pi_L : V \rightarrow L$ the orthogonal projection onto L . Then, $\Gamma_{LH}(\varphi) = \tau$. The second statement in the Proposition follows from the definition of Γ_{LH} . \square

This section concludes with an example that illustrates the conclusion of the Instrumental Lemma.

Example 5.2.9. *Let $Q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ denote the quadratic map given by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 - x_2^2 \\ 0 \\ x_1^2 + x_2^2 \end{pmatrix}$$

Q is equivalent to the quadratic map,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 \\ x_2^2 \\ 0 \end{pmatrix} \tag{5.2.1}$$

The quadratic map in equation (5.2.1) is convex. It follows that Q is convex as well. The Instrumental Lemma implies that there must be quadratic maps arbitrarily close to Q in $\mathcal{Q}(\mathbb{R}^2, \mathbb{R}^3)$ that are not convex. This example produces such a map by the method suggested in the proof of the lemma.

The proof first asks whether Q is nondegenerate. If Q fails to be nondegenerate, the proof uses a nondegenerate map that is arbitrarily close to Q .

The associated linear map $\psi : \mathbb{R}^3 \rightarrow \text{Sym}_2$ of Q is given by

$$\psi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \psi(e_2) = 0 \quad \psi(e_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that Q is not nondegenerate. However, if $\varepsilon \neq 0$, the map ψ_ε given by,

$$\psi_\varepsilon(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \psi_\varepsilon(e_2) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \quad \psi_\varepsilon(e_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is nondegenerate. Furthermore, the family of maps $\{\psi_\varepsilon \mid \varepsilon \in \mathbb{R}, \varepsilon \neq 0\}$ comes arbitrarily close to ψ in $\mathcal{L}(\mathbb{R}^3, \text{Sym}_2)$. To prove that Q is not stably convex, it therefore suffices to prove that the quadratic map Q_ε associated with ψ_ε is not convex if $\varepsilon \neq 0$. Note that Q_ε is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 - x_2^2 \\ 2\varepsilon x_1 x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$$

When $\varepsilon \neq 0$, ψ_ε is injective. The proof of the Instrumental Lemma then implies that Q_ε is not convex. To see this explicitly, note that because $e_3 \in \text{Int}_{\mathbb{R}^3}(K_\psi)$, $H_{e_3}^\Gamma \subset \mathbb{R}^3$ is

a flat base for Q_ε . But,

$$H_{e_3}^1 = \left\{ \begin{pmatrix} \cos(\theta) \\ \varepsilon \sin(\theta) \\ 1 \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

which is not convex. Q_ε is therefore not convex by Lemma 3.4.17.

5.3 A PROOF OF THE ROUNDNESS THEOREM: THE FORWARD IMPLICATION

This section proves the forward implication in the Roundness Theorem: that every definite quadratic map in $\mathcal{Q}_{n,m}$ (with $4 \leq m \leq n$) that is not round is not stably convex. There are two fixed ingredients in the proof: a definite quadratic map $Q \in \mathcal{Q}_{n,m}$ that is not round (with ALM ψ), and an open neighbourhood $U \subset \mathcal{Q}_{n,m}$ of Q . The task at hand is this: *find a quadratic map $Q' \in U$ that is not convex.*

Underlying the proof is a cascade of simple observations, each of which has roots in Proposition 4.2.17. Let $Q' \in \mathcal{Q}_{n,m}$ be any quadratic map, with associated linear map ψ' . Then,

- Q' is convex if and only if $u^\perp \cap \text{Im } Q'$ is convex, for all $u \in \mathbb{R}^m$.
- When $u \in K_\psi$, $u^\perp \cap \text{Im } Q'$ is itself the image of a quadratic map, namely $Q'_u := \pi_{u^\perp} \circ Q' \circ i_{\ker \psi'(u)} : \ker \psi'(u) \rightarrow u^\perp$, where $\pi_{u^\perp} : \mathbb{R}^m \rightarrow u^\perp$ is the orthogonal projection onto u^\perp , and $i_{\psi'(u)} : \ker \psi'(u) \rightarrow \mathbb{R}^n$ the inclusion map of $\ker \psi'(u)$ into \mathbb{R}^n .
- Q' is convex only if Q'_u is convex, for all $u \in K_\psi$

The forthcoming proof is guided by this conceit: when Q is definite, but not round, there exists a $u \in K_\psi \subset \mathbb{R}^m$ for which $Q_u := \pi_{u^\perp} \circ Q \circ i_{\ker \psi(u)}$ is not stably convex. Thus, either Q_u is not convex, in which case $u^\perp \cap \text{Im } Q$ and

therefore Q itself are not either, or Q_u is convex, but only delicately so. In the event that Q_u is convex, its unstable convexity nonetheless makes it possible to find a quadratic map $\widehat{Q} \in U$ for which $u^\perp \cap \text{Im } \widehat{Q}$ is not convex. In either case, Q is not stably convex. Briefly, the proof works by deforming the face $u^\perp \cap \text{Im } Q$ so that it is not convex.

The proof thus consists of two steps: first, finding $u \in K_\psi$ for which Q_u is not stably convex, and second, using Q_u to construct a nonconvex $\widehat{Q} \in U$.

STEP 1: FINDING $u \in K_\psi$ FOR WHICH Q_u IS NOT STABLY CONVEX

Because Q is definite but not round, there exists a $u \in K_\psi$ for which $1 \leq \text{rank } \psi(u) < n - 1$. An additional assumption must be made here.

Assumption 5.3.1. *There exists a $u \in K_\psi$ for which $\text{rank } \psi(u) = n - 2$.*

This is an important assumption that facilitates the proof of the forward implication; it will be revisited once the proof is complete (see Section 5.3.1 below). Assumption 5.3.1 does not affect the reach of the forthcoming proof; it will still follow, albeit with some additional effort, that *no* nonround quadratic map is stably convex.

Remark 5.3.2. Henceforth, $u \in K_\psi \subset \mathbb{R}^m$ will always denote the element posited by Assumption 5.3.1, and $M_u \in \text{Sym}_n$ will denote $\psi(u)$ (there will be a number of ψ 's involved in the following discussion, some primed and some with hats; having a separate symbol for $\psi(u)$ should make things clearer). Finally, given a quadratic map $Q' \in \mathcal{Q}_{n,m}$, Q'_u will denote the map $\pi_{u^\perp} \circ Q' \circ i_{\ker M_u}$, which maps from $\ker M_u$ to u^\perp .

Proposition 5.3.3. *Q_u is not stably convex.*

Proof. Assumption 5.3.1 implies that $\dim \ker M_u = 2$. Thus, $Q_u : \ker M_u \rightarrow u^\perp$ maps from a 2-dimensional space to a space, $u^\perp \subset \mathbb{R}^m$, whose dimension is strictly greater than 2 (recall that $m \geq 4$). Q_u , therefore, is subject to the Instrumental Lemma; *it is not stably convex.* \square

STEP 1.5: A HELPFUL REMARK

The map taking $Q' \in \mathcal{Q}_{n,m}$ to $Q'_u \in \mathcal{Q}(\ker M_u, u^\perp)$ will play an important role in the construction of a nonconvex $\widehat{Q} \in U$. Of particular significance is the relationship between the ALM of Q'_u (namely, ψ'_u) and that of Q' (namely, ψ'). Their relationship is described in Proposition 5.2.5, which says that

$$\psi'_u = R_{\ker M_u} \circ (\psi')|_{u^\perp} \quad (5.3.1)$$

where $R_{\ker M_u}$ is the restriction map defined in Remark 5.2.4.

For the purposes of this section, the most important aspect of (5.3.1) is this: ψ'_u depends only on the values that ψ' takes on u^\perp . Applied to Q and its ALM ψ , this raises the prospect of modifying the intersection $u^\perp \cap \text{Im } Q (= \text{Im } Q_u)$ by changing only the values that ψ takes on u^\perp , all the while keeping $\psi(u)$ fixed (fixed at M_u).

STEP 2: CONSTRUCTING A NONCONVEX $\widehat{Q} \in U$

Proposition 5.3.3 says that in any given neighbourhood of Q_u in $\mathcal{Q}(\ker M_u, u^\perp)$, there exists a nonconvex map $P : \ker M_u \rightarrow u^\perp$. The strategy underlying the construction of \widehat{Q} is to arrange for the intersection $u^\perp \cap \text{Im } \widehat{Q}$ to be equal to $\text{Im } P$ for some such P , and to thereby be nonconvex.

A good place to start looking for \widehat{Q} is Q itself. After all, $u^\perp \cap \text{Im } Q = \text{Im } Q_u$, and Q_u is just barely convex if it is at all. Step 1.5 above hints at an approach to modifying Q in such a way that the resulting map \widehat{Q} has a nonconvex intersection $u^\perp \cap \text{Im } \widehat{Q}$ with u^\perp , namely by modifying the restriction $\psi|_{u^\perp}$ while keeping $\psi(u)$ fixed. If \widehat{Q} is obtained from Q in this manner, it, and its ALM $\widehat{\psi}$ will satisfy

- $\widehat{\psi}(u) = M_u$
- $u^\perp \cap \text{Im } \widehat{Q} = \text{Im } \widehat{Q}_u$

Furthermore, $\widehat{Q}_u : \ker M_u \rightarrow u^\perp$ will be the quadratic map whose ALM is, according to (5.3.1) above, $R_{\ker M_u} \circ (\widehat{\psi}|_{u^\perp})$. The question, then, is this: is it

possible to find a linear map $\widehat{\psi} \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ for which $R_{\ker M_u} \circ (\widehat{\psi}|_{u^\perp})$ the ALM of a nonconvex map? The following proposition gives an affirmative answer.

Proposition 5.3.4. *There exists a nonconvex $\widehat{Q} \in U$.*

Remark 5.3.5. Recall that the fundamental relation (4.1.2) defines an isomorphism between $\mathcal{Q}_{n,m}$ and $\mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ (and, more generally, between $\mathcal{Q}(V, W)$ and $\mathcal{L}(W, \text{Sym}(V))$ for any spaces V and W). The identification of $\mathcal{Q}_{n,m}$ with $\mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ will be used without notice in the proof of Proposition 5.3.4. For example, the neighbourhood U will refer both to an open neighbourhood of Q in $\mathcal{Q}_{n,m}$, and to the corresponding open neighbourhood of ψ in $\mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$. ■

A simple lemma is needed. It says nothing more than that a linear map in $\mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ is defined entirely by its value at u , and by its restriction to u^\perp . However, it helps to formalize the notion of “only modifying $\psi|_{u^\perp}$ ”.

Lemma 5.3.6. *The map from $\mathcal{L}(\mathbb{R}^n, \text{Sym}_n)$ to $\text{Sym}_n \oplus \mathcal{L}(u^\perp, \text{Sym}_n)$ taking ψ' to $(\psi'(u), \psi'|_{u^\perp})$ is an isomorphism.*

Proof. (Proof of Proposition 5.3.4) Under the identification described in Lemma 5.3.6, the neighbourhood U contains a product neighbourhood $U_1 \times U_2$, where $U_1 \subset \text{Sym}_n$ is an open neighbourhood of M_u , and $U_2 \subset \mathcal{L}(u^\perp, \text{Sym}_n)$ is an open neighbourhood of $\psi|_{u^\perp}$. To prove Proposition 5.3.4, it suffices to find a nonconvex quadratic map in $U_1 \times U_2$ (or, more precisely, the ALM of a nonconvex map).

To that end, consider the map $\Gamma : \mathcal{L}(u^\perp, \text{Sym}_n) \rightarrow \mathcal{L}(u^\perp, \text{Sym}(\ker M_u))$ taking ψ' to $R_{\ker M_u} \circ \psi'$. Γ has two important properties (the first of which can be found in Proposition 5.2.5),

- Γ is a linear surjection, and therefore an open map.
- $\Gamma(\psi|_{u^\perp}) = R_{\ker M_u} \circ (\psi|_{u^\perp})$, which, by (5.3.1), is the ALM of the unstably convex map Q_u .

It follows that $\Gamma(U_2)$ is an open neighbourhood of the ALM, $R_{\ker M_u} \circ (\psi|_{u^\perp})$, of Q_u . Because Q_u is not stably convex, there exists a nonconvex quadratic map $P : \ker M_u \rightarrow u^\perp$ whose ALM lies in $\Gamma(U_2)$. By virtue of lying in $\Gamma(U_2)$, the ALM of P is of the form $R_{\ker M_u} \circ \widehat{\widehat{\psi}}$ for some $\widehat{\widehat{\psi}} \in U_2$ (there are two hats on $\widehat{\widehat{\psi}}$ because $\widehat{\psi}$ was designated as the ALM of \widehat{Q} earlier).

Let $\widehat{Q} \in \mathcal{Q}_{n,m}$ be the quadratic map whose ALM is $(M_u, \widehat{\widehat{\psi}}) \in \text{Sym}_n \oplus \mathcal{L}(u^\perp, \text{Sym}_n)$. Then, \widehat{Q} lies in $U_1 \times U_2 \subset U$ and is not convex, since $u^\perp \cap \text{Im } \widehat{Q} = \text{Im } P$. □

5.3.1 NOT SO FAST: A WORD ON ASSUMPTION 5.3.1

Hidden in the proof of Proposition 5.3.4 is Assumption 5.3.1; the proof shows only that the proposition is true for those $Q \in \mathcal{Q}_{n,m}$ for which there exists a $u \in K_\psi$ with $\text{rank } M_u = n - 2$. Fortunately, this turns out to be sufficient to conclude that Proposition 5.3.4 is true *for every* nonround quadratic map in $\mathcal{Q}_{n,m}$. The forward implication in the Roundness Theorem still stands.

To simplify the notation of the forthcoming discussion, let $\text{GNR}_{n,m} \subset \mathcal{Q}_{n,m}$ denote the subset defined by Assumption 5.3.1,

Definition 5.3.7. *The subset $\text{GNR}_{n,m} \subset \mathcal{Q}_{n,m}$ defined by*

$$\text{GNR}_{n,m} = \{ \psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n) \mid \exists u \in K_\psi \text{ s.t. } \psi(u) = n - 2 \}$$

*is the set of **generically nonround quadratic maps**. If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, $\text{GNR}(V, W)$ will be written as $\text{GNR}_{n,m}$.*

Remark 5.3.8. Definition 5.3.7 also uses the identification of $\mathcal{Q}_{n,m}$ with $\mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$. As a subset of $\mathcal{Q}_{n,m}$, $\text{GNR}_{n,m}$ is to be understood as the set of quadratic maps whose ALMS lie in the set defined in Definition 5.3.7. ■

To see why Assumption 5.3.1 is not too restrictive, note that what the forward implication in the Roundness Theorem is saying is this

$$\text{NR}_{n,m} \cap \text{Def}_{n,m} \cap \text{Int}_{\mathcal{Q}_{n,m}}(\text{Convex}_{n,m}) = \emptyset$$

That is, the set of definite, nonround round quadratic maps $\text{NR}_{n,m} \cap \text{Def}_{n,m}$ does not intersect the interior of the set of convex quadratic maps; nonround definite maps are not stably convex, in other words.

In general, to show that two subsets, A and B , of a topological space X satisfy $A \cap \text{Int}_X(B) = \emptyset$, it suffices to find a dense subset of A , $D \subset A$, that satisfies $D \cap \text{Int}_X(B) = \emptyset$. As it turns out, $\text{GNR}_{n,m} \cap \text{Def}_{n,m}$ is dense in $\text{NR}_{n,m} \cap \text{Def}_{n,m}$.

Proposition 5.3.9. *$\text{GNR}_{n,m} \cap \text{Def}_{n,m}$ is a dense subset of $\text{NR}_{n,m} \cap \text{Def}_{n,m}$.*

Proof. Let $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ be the ALM of a nonround, definite quadratic map, and let $u \in \mathbb{R}^m$ be such that $u \in K_\psi$, and $\text{rank } \psi(u) < n - 2$. Finally, let $U_1 \times U_2 \subset \text{Sym}_n \oplus \mathcal{L}(u^\perp, \text{Sym}_n)$ be open an open neighbourhood of $(\psi(u), \psi|_{u^\perp})$, where U_1 is an open neighbourhood of $\psi(u)$ in Sym_n , and U_2 an open neighbourhood of $\psi|_{u^\perp}$ in $\mathcal{L}(u^\perp, \text{Sym}_n)$. Within U_1 is a rank $n - 1$ matrix M . Then, the quadratic map whose ALM is $(M, \psi|_{u^\perp}) \in \text{Sym}_n \oplus \mathcal{L}(u^\perp, \text{Sym}_n)$ is generically nonround. \square

5.3.2 AN EXAMPLE

This section concludes with an example that demonstrates the principles of the proof of the forward implication.

Example 5.3.10. *This example shows that the standard quadratic map in 4 dimensions is not stably convex. The method used here can be applied to the standard quadratic map in any dimension. Throughout this example, \mathbb{R}^k (for $k = 1, 2, 3$) will be thought of as the subspace of \mathbb{R}^4 consisting of those points whose last k coordinates vanish*

Actually, the example demonstrates the unstable convexity of a map that is equivalent (in the sense of Section 4.6) to the standard quadratic map. The map in question is the map $Q \in \mathcal{Q}_{4,4}$ whose defining matrices are

$$\begin{aligned}
M_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & M_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
M_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & M_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Explicitly,

$$Q : \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \\ x_3^2 - x_4^2 \\ x_3^2 + x_4^2 \end{pmatrix}$$

Q is equivalent to the standard quadratic map in $\mathcal{Q}_{4,4}$ (see Remark 4.6.4).

Let $U \subset \mathcal{Q}_{4,4}$ be a fixed, open neighbourhood of Q . To prove that Q is not stably convex, it is necessary to find a nonconvex quadratic map lying in U .

The first step in the proof above was to find a $u \in K_\psi$ such that $\text{rank } \psi(u) = n - 2$ ($= 2$ in this case). There are many such u . One choice is $u = e_4$, since $\psi(e_4) \in S_n^+$ and $\text{rank } \psi(e_4) = 2$.

The next step in the proof was to consider the map $Q_u := \pi_{e_n^\perp} \circ (Q|_{\ker \psi(u)}) : \ker \psi(u) \rightarrow u^\perp$. For $u = e_4$, $\ker \psi(u) = \ker \psi(e_4) = \mathbb{R}^2$, $u^\perp = e_4^\perp = \mathbb{R}^3$, and Q_u is given by

$$Q_{e_4} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \\ 0 \end{pmatrix} \quad (5.3.2)$$

Q_{e_4} is not stably convex. This was demonstrated in Example 5.2.9, which showed that for every $\varepsilon \neq 0$, the quadratic map $P_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$P_\varepsilon : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \\ 2\varepsilon x_1 x_2 \end{pmatrix} \quad (5.3.3)$$

is not convex. Moreover, by a judicious choice of ε , P_ε can be made arbitrarily close to Q_{e_4} in $\mathcal{Q}_{2,3}$.

The penultimate step in the proof was to construct find a linear map $\widehat{\widehat{\psi}} \in \mathcal{L}(u^\perp, \text{Sym}_n)$ lying sufficiently close to $\psi|_{u^\perp}$ and satisfying the following property,

- $\Gamma(\widehat{\widehat{\psi}}) := R_{\ker \psi(u)} \circ \widehat{\widehat{\psi}}$ is the ALM of a nonconvex quadratic map P that is sufficiently close to Q_u .

An obvious choice for P is P_ε from (5.3.3), which can be made arbitrarily close to Q_u by properly choosing ε .

In the present case, the map Γ maps from $\mathcal{L}(e_4^\perp, \text{Sym}_4) = \mathcal{L}(\mathbb{R}^3, \text{Sym}_4)$ to $\mathcal{L}(e_4^\perp, \text{Sym}(\ker \psi(e_4))) = \mathcal{L}(\mathbb{R}^3, \text{Sym}_2)$. For $\widehat{\widehat{\psi}} \in \mathcal{L}(e_4^\perp, \text{Sym}_4)$, $(\Gamma(\widehat{\widehat{\psi}}))(u) \in \text{Sym}_2$ is simply the upper 2×2 block of $\widehat{\widehat{\psi}}(u) \in \text{Sym}_4$, for $u \in \mathbb{R}^3$.

Thus, if $\Gamma(\widehat{\widehat{\psi}})$ is to be the ALM of P_ε , it must satisfy these equations,

$$\text{The upper } 2 \times 2 \text{ block of } \widehat{\widehat{\psi}}(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{The upper } 2 \times 2 \text{ block of } \widehat{\widehat{\psi}}(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{The upper } 2 \times 2 \text{ block of } \widehat{\widehat{\psi}}(e_3) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

$\widehat{\widehat{\psi}}$ can be ensured to be sufficiently close to $\psi|_{e_4^\perp}$ by choosing the remaining entries of $\widehat{\widehat{\psi}}(e_i) \in \text{Sym}_4$ to equal those of $(\psi|_{e_4^\perp})(e_i)$, where $i = 1, 2, 3$. Thus,

$$\widehat{\widehat{\psi}}(e_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \widehat{\widehat{\psi}}(e_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \widehat{\widehat{\psi}}(e_3) = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Finally, the quadratic map $\widehat{Q} \in \mathcal{Q}_{4,4}$ is ALM is $(\psi(e_4), \widehat{\widehat{\psi}}) \in \text{Sym}_4 \oplus \mathcal{L}(\mathbb{R}^3, \text{Sym}_4)$ is not convex. That map is,

$$Q' : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \\ 2\varepsilon x_1 x_2 + x_3^2 - x_4^2 \\ x_3^2 + x_4^2 \end{pmatrix}$$

It is plain to see that $\text{Im } Q' \cap \mathbb{R}^3$ is not convex; it is equal to $\text{Im } P_\varepsilon$.

Remark 5.3.11. If the map \widehat{Q} in Example 5.3.10 is postcomposed with the matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

then the resulting (nonconvex) quadratic map is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 \\ x_2^2 \\ \varepsilon x_1 x_2 + x_3^2 \\ -\varepsilon x_1 x_2 + x_4^2 \end{pmatrix}$$

This provides a more vivid demonstration of the unstable convexity of the standard quadratic map. ■

5.4 A PROOF OF ROUNDNESS THEOREM: THE BACKWARD IMPLICATION

This section proves that all round quadratic maps in $\mathcal{Q}_{n,m}$ (with $m \geq 4$, $n \geq m$, $n \neq m+1$) are convex; equivalently, $\text{Round}_{n,m} \subset \mathcal{Q}_{n,m}$ is contained in $\text{Convex}_{n,m} \subset \mathcal{Q}_{n,m}$. Given that $\text{Round}_{n,m}$ is an open subset of $\mathcal{Q}_{n,m}$ (see Section 4.5.2), the containment $\text{Round}_{n,m} \subset \text{Convex}_{n,m}$ will show that round maps are stably convex and will thereby complete the proof of the Roundness Theorem. Henceforth n and m will denote natural numbers satisfying the restrictions in the Roundness Theorem, $Q \in \mathcal{Q}_{n,m}$ will denote a round quadratic map, and $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ will denote its ALM.

Recall, from Section 4.6, that a quadratic map is convex if and only if every map to which it is equivalent is also convex. Section 4.6 also pointed out that

every definite quadratic map in $\mathcal{Q}_{n,m}$ is equivalent to a map having the form

$$x \in \mathbb{R}^n \mapsto \begin{pmatrix} \langle x, M_1 x \rangle \\ \vdots \\ \langle x, M_{m-1} x \rangle \\ ||x||^2 \end{pmatrix} \quad (5.4.1)$$

That is, every definite map in $\mathcal{Q}_{n,m}$ is equivalent to one whose last defining matrix $M_m \in \text{Sym}_n$ is equal to the identity map on \mathbb{R}^n . Thus, to prove the backwards implication of the Roundness Theorem, it suffices to show that every round quadratic map with the form (5.4.1) is convex. It will henceforth be assumed that $\psi(e_m) = \mathbb{I}_n$ (and that $\psi_{e_i} = M_i$ for some $M_i \in \text{Sym}_n$, for $i = 1, \dots, m-1$).

The assumption that $\psi(e_m) = \mathbb{I}_n$ implies that $e_m \in \text{Int}_{\mathbb{R}^m} K_\psi$. It follows that $H_{e_m}^1 \cap \text{Im } Q \subset \mathbb{R}^n$ is a flat base for $\text{Im } Q$ (see Lemma 3.4.14, and recall that $H_{e_m}^1$ is the affine hyperplane in \mathbb{R}^m consisting of those points $x \in \mathbb{R}^m$ that satisfy $\langle x, e_m \rangle_{\mathbb{R}^m} = 1$). Let $B \subset \mathbb{R}^n$ denote the flat base $H_{e_m}^1 \cap \text{Im } Q$. Then, Q is convex if and only if B is convex.

The form (5.4.1) that Q possesses implies that $Q(x) \in B$ if and only if $x \in \mathbb{S}^{n-1}$. Thus,

$$B = \left\{ \begin{pmatrix} \langle x, M_1 x \rangle \\ \vdots \\ \langle x, M_{m-1} x \rangle \\ 1 \end{pmatrix} : x \in \mathbb{S}^{n-1} \right\} \quad (5.4.2)$$

Let $Q_S : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^{n-1}$ be the map

$$Q_S : x \in \mathbb{S}^{n-1} \mapsto \begin{pmatrix} \langle x, M_1 x \rangle \\ \vdots \\ \langle x, M_{m-1} x \rangle \end{pmatrix}$$

That is, $Q_S = \pi_{e_m^\perp} \circ Q|_{\mathbb{S}^{n-1}}$, where $\pi_{e_m^\perp} : \mathbb{R}^n \rightarrow e_m^\perp = \mathbb{R}^{m-1}$ is the orthogonal

projection onto e_m^\perp , which is identified with \mathbb{R}^{m-1} in the usual way.

If the affine hyperplane $H_{e_m}^1$ is also identified with \mathbb{R}^{m-1} , then, as subsets of \mathbb{R}^{m-1} , the image of Q_S is equal to the flat base B . It follows that B , and therefore Q is convex if and only if the image of Q_S is convex. This warrants a definition.

Defintion 5.4.1. *Let $Q \in \mathcal{Q}_{n,m}$ be a quadratic map. The restriction $Q|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^m$ is **convex** if its image is a convex subset of \mathbb{R}^m .*

The remainder of this section is devoted to showing that Q_S is convex whenever Q is round. Q_S has three properties that, together with the help of an adjunct lemma, imply that it is convex.

Property 1: $\text{Conv}(\text{Im } Q_S)$ is a convex body in \mathbb{R}^{m-1} .

Property 2: $Q_S^{-1}(\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)) = E \coprod -E$, where $E \subset \mathbb{S}^{n-1}$ is a smoothly embedded copy of \mathbb{S}^{m-2} and $-E$ its image under the antipodal map of \mathbb{S}^{n-1} (\coprod denotes the disjoint union).

Property 3: The restriction of Q_S to E is a homeomorphism between E and $\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)$.

The proofs of Properties 1, 2 and 3 will be provided shortly. The fact that they imply the convexity of Q_S deserves immediate attention. The forthcoming discussion will be simpler if Q_S is thought of as mapping into $\text{Conv}(\text{Im } Q_S)$, $Q_S : \mathbb{S}^{n-1} \rightarrow \text{Conv}(\text{Im } Q_S)$. The question of the convexity of Q_S then becomes this: is Q_S surjective?

Property 1 implies that there exists a homeomorphism $h : \text{Conv}(\text{Im } Q_S) \rightarrow \mathbb{D}^{m-1}$, where \mathbb{D}^{m-1} is the closed unit disk in \mathbb{R}^{m-1} . Let $F : \mathbb{S}^{n-1} \rightarrow \mathbb{D}^{m-1}$ denote the composition $F = h \circ Q_S$. Then, Q_S is surjective if and only if F is surjective. Moreover, by virtue of Properties 2 and 3, F itself has the following properties (in which \mathbb{S}^{m-2} should be thought of as the boundary of \mathbb{D}^{m-1}),

- $F^{-1}(\mathbb{S}^{m-2}) = E \coprod -E$, where $E \subset \mathbb{S}^{n-1}$ is a smoothly embedded copy of \mathbb{S}^{m-2} .
- $F|_E : E \rightarrow \mathbb{S}^{m-2}$ is a homeomorphism.

As it turns out, *any* continuous map from \mathbb{S}^{n-1} to \mathbb{D}^{m-1} (with $m \geq 4$, $n \geq m$, $n \neq m + 1$) satisfying these two properties is surjective. This is the adjunct lemma mentioned above.

Lemma 5.4.2. (The Surjectivity Lemma) Suppose that $m \geq 4$, $n \geq m$, $n \neq m + 1$. Let $F : \mathbb{S}^{n-1} \rightarrow \mathbb{D}^{m-1}$ be a continuous function with the following properties,

- $F^{-1}(\mathbb{S}^{m-2}) = E_1 \coprod E_2$ where each E_i is diffeomorphic to \mathbb{S}^{m-2} .
- For $i = 1, 2$, $F|_{E_i} : E_i \rightarrow \mathbb{S}^{m-2}$ is a homeomorphism.

Then, F is surjective.

The proofs of Properties 1 through 3, together with the proof of Theorem 5.4.2, will complete the proof of the Roundness Theorem.

A PROOF OF PROPERTY 1

To see that $\text{Conv}(\text{Im } Q_S)$ is a convex body in \mathbb{R}^{m-1} , note first that, since $\text{Im } Q_S$ compact, $\text{Conv}(\text{Im } Q_S)$ is compact. Moreover, because Q is a round quadratic map, $\text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q)) \neq \emptyset$. It follows that $\text{Int}_{H_{e_m}} B \neq \emptyset$ (see Lemma 3.4.21), and therefore that $\text{Int}_{\mathbb{R}^{m-1}}(\text{Conv}(\text{Im } Q_S)) \neq \emptyset$, since $\text{Im } Q_S = B$ (under the identification of $H_{e_m}^\perp$ with $e_m^\perp = \mathbb{R}^{m-1}$).

A PROOF OF PROPERTY 2

Property 2 is partly a consequence of what was observed in Section 4.5.3: when Q is round, the boundary of K_ψ (apart from $0 \in \mathbb{R}^m$) is an embedded submanifold of \mathbb{R}^m , and the boundary of any flat base of K_ψ is diffeomorphic to \mathbb{S}^{m-2} . The following proof of Property 2 shows that $Q_S^{-1}(\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S))$ is a double covering of the boundary of any given flat base of K_ψ . This will complete the proof, since the only double covering of \mathbb{S}^{m-2} is the disconnected one, $\mathbb{S}^{m-2} \times \{0, 1\}$.

Section 4.4.4 established that if $Q \in \mathcal{Q}_{n,m}$ is a definite quadratic map,

$$Q^{-1}(\partial_{\mathbb{R}^m} \text{Conv}(\text{Im } Q)) = \bigcup_{u \in \partial_{\mathbb{R}^m} K_\psi - \{0\}} \ker \psi(u)$$

It follows that

$$Q_S^{-1}(\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)) = \mathbb{S}^{n-1} \cap \bigcup_{u \in \partial_{\mathbb{R}^m} K_\psi - \{0\}} \ker \psi(u) \quad (5.4.3)$$

The forthcoming proof of Property 2 rests on a certain interpretation of Equation (5.4.3).

Because Q is round, the union in the right hand side of Equation (5.4.3) is a union of 1-dimensional subspaces of \mathbb{R}^n . It follows that the preimage $Q_S^{-1}(\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S))$ is actually given by,

$$Q_S^{-1}(\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)) = \pi^{-1}(U)$$

where $\pi : \mathbb{S}^{n-1} \rightarrow \mathbb{RP}^{n-1}$ is the standard covering map and $U \subset \mathbb{RP}^{n-1}$ the subset

$$U = \{\ker \psi(u) : u \in \partial_{\mathbb{R}^m} K_\psi - \{0\}\}$$

To prove Property 2, it now suffices to show that $U \subset \mathbb{RP}^{n-1}$ is diffeomorphic to \mathbb{S}^{m-2} . For, in that case, Lemma 3.8.8 says that $\pi^{-1}(U) = E \amalg -E$, with E diffeomorphic to \mathbb{S}^{m-2} . The remainder of this discussion is devoted to showing that U is an embedded copy of \mathbb{S}^{m-2} .

A careful look at U will draw the proof to its conclusion. To that end, let $H \subset \mathbb{R}^m$ be an affine hyperplane which is such that $H \cap K_\psi$ is a flat base for K_ψ . Then, every $u \in \partial_{\mathbb{R}^m} K_\psi - \{0\}$ is a multiple of some $u' \in \partial_H(H \cap K_\psi)$ (see

Lemma 3.4.21). It follows that $U \subset \mathbb{RP}^{n-1}$ can be written as this,

$$U = \{\ker \psi(u) : u \in \partial_H(H \cap K_\psi)\}$$

The definition of U can be reexpressed still. Let $S \subset H$ denote the boundary of the flat base $H \cap K_\psi$, $S = \partial_H(H \cap K_\psi)$ (recall that S is diffeomorphic to \mathbb{S}^{m-2}), and let $b_\psi : S \rightarrow \mathbb{RP}^{n-1}$ denote the map $u \in S \mapsto \ker \psi(u) \in \mathbb{RP}^{n-1}$. Then,

$$U = \text{Im } b_\psi$$

Thus, to show that U is diffeomorphic to \mathbb{S}^{m-2} , it suffices to show that b_ψ is an embedding of S into \mathbb{RP}^{n-1} . Because S is compact, b_ψ is an embedding so long as it is an injective immersion. The following claim thereby completes the proof of Property 2.

Claim 5.4.3. $b_\psi : S \rightarrow \mathbb{RP}^{n-1}$ is an injective immersion.

Proof. Proposition 4.5.17 proved that $\ker \psi(u_1) \neq \ker \psi(u_2)$ so long as $u_1, u_2 \in \partial_{\mathbb{R}^m} K_\psi$ are linearly independent. But, if distinct u_1 and u_2 both lie in the flat base $H \cap K_\psi$ (and in particular, if they lie in the boundary S), they are necessarily linearly independent (see Remark 3.4.16). Thus, b_ψ is injective.

Show that b_ψ is an immersion requires an earlier result:

- The kernel of the Jacobian of the map $k : S_{n,n-1}^+ \rightarrow \mathbb{RP}^{n-1}$ mapping $M \mapsto \ker M$ is,

$$\ker k_{*,M} = \{BM + MB^T : B \in \mathfrak{gl}_n, \ker M \subset \ker B\} \subset \text{Sym}_n$$

(see Proposition 3.8.15).

The map b_ψ first takes a $u \in S$ to $\psi(u) \in S_{n,n-1}^+$, and then to $\ker \psi(u) \in \mathbb{RP}^{n-1}$. Thus, b_ψ is actually the composition of two maps: $b_\psi = k \circ (\psi|_S)$ (recall that $S = \partial_H(H \cap K_\psi)$ is the boundary of the flat base $H \cap K_\psi$ of K_ψ).

Because it is the ALM of a round map, ψ is injective. Thus, to prove that b_ψ is an immersion, it suffices to prove that $\psi(T_u \partial_{\mathbb{R}^m} K_\psi) \cap \ker k_{*,\psi(u)} = \{0\} \subset \text{Sym}_n$ for all $u \in S$.

Suppose, then, that there exists a $u \in S$ and a nonzero $u' \in T_u(\partial_{\mathbb{R}^m} K_\psi - \{0\})$ for which $\psi(u') \in \ker k_{*,\psi(u)}$. Note that by virtue of being in $T_u(\partial_{\mathbb{R}^m} K_\psi - \{0\})$, u' is linearly independent of u . It would follow from the description of $\ker k_{*,\psi(u)}$ just given, that $\ker \psi(u) \subset \ker \psi(u')$. However, this very possibility was forbidden by Proposition 4.5.19. \square

Remark 5.4.4. Property 2 is not true when $m \leq 3$: the preimage $Q_S^{-1}(\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S))$ need not be the disjoint union of two copies of \mathbb{S}^{m-2} when $m \leq 3$. For example, when $Q : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the following quadratic map,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \\ x_1^2 + x_2^2 + x_3^2 \end{pmatrix}$$

the preimage $Q_S^{-1}(\partial_{\mathbb{R}^2} \text{Conv}(\text{Im } Q_S))$ is equal to the single equatorial copy of \mathbb{S}^1 in \mathbb{S}^2 . The reason for this is that \mathbb{S}^1 has nontrivial connected covering spaces. Indeed, the restriction of Q_S to $Q_S^{-1}(\partial_{\mathbb{R}^2} \text{Conv}(\text{Im } Q_S))$ is equivalent to the double covering $e^{i\theta} \mapsto e^{i2\theta}$ of \mathbb{S}^1 . ■

A PROOF OF PROPERTY 3

Since E is compact, $Q_S|_E$ is a homeomorphism between E and $\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)$ so long as it is a bijection.

To see that $Q_S|_E$ is surjective, recall that under the identification of $H_{e_m}^1$ with $e_m^\perp = \mathbb{R}^{m-1}$, the image of Q_S is the same as the base B of (5.4.2). Thus, under that same identification, $\partial_{H_{e_m}^1} \text{Conv}(B) = \partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)$.

By Proposition 3.4.19, the boundary $\partial_{H_{e_m}^1} \text{Conv}(B)$ of $\text{Conv}(B)$ is equal to the intersection $H_{e_m}^1 \cap \partial_{\mathbb{R}^m} \text{Conv}(\text{Im } Q)$. But, because Q is round, it satisfies the set containment $\partial_{\mathbb{R}^m} \text{Conv}(\text{Im } Q) \subset \text{Im } Q$ of Proposition 4.5.10. It follows at last

that $H_{e_m}^1 \cap \partial_{\mathbb{R}^m} \text{Conv}(\text{Im } Q) \subset \text{Im } Q$, and therefore that $\partial_{\mathbb{R}^{m+1}} \text{Conv}(\text{Im } Q_S) \subset \text{Im } Q_S$; $Q_S|_E$ is surjective.

To see that $Q_S|_E$ is injective, note that E consists of exactly one point from each of the intersections $\mathbb{S}^{n-1} \cap \ker \psi(u)$, as u varies over $\partial_{\mathbb{R}^m} K_\psi - \{0\}$. Suppose now that $Q(x_1) = Q(x_2)$ for some $x_1, x_2 \in E$, with $x_i \in \ker \psi(u_i)$, $i = 1, 2$, for some nonzero $u_1, u_2 \in \partial_{\mathbb{R}^m} K_\psi - \{0\}$. Thus, $Q(\ker \psi(u_1)) = Q(\ker \psi(u_2))$. But then, by Proposition 4.5.18, $\ker \psi(u_1) = \ker \psi(u_2)$, and therefore, $x_1 = x_2$. $Q_S|_E$ is injective.

A PROOF OF THE SURJECTIVITY LEMMA

By virtue of the embedding theorems in Section 3.8.7, each of the hypothetical embedded copies, E_1 and E_2 , of \mathbb{S}^m , bounds an $m + 1$ -dimensional disk in \mathbb{S}^{n-1} . Let $D_1 \subset \mathbb{S}^{n-1}$ be an $m + 1$ -dimensional disk bounded by E_1 . The Surjectivity Lemma follows from a careful inspection of the restriction $F|_{D_1} : D_1 \rightarrow \mathbb{D}^{m+1}$ of F to D_1 . Such an inspection will find that $F|_{D_1}$ is itself surjective; this trivially implies the surjectivity of F .

For clarity, let \bar{F} denote the restriction $F|_{D_1}$. The most important property of \bar{F} is this: thought of as a map into \mathbb{S}^m , the restriction $\bar{F}|_{E_1} : E_1 \rightarrow \mathbb{S}^m$ of \bar{F} to $E_1 \subset D_1$ is a homeomorphism. The surjectivity of \bar{F} now follows from the next proposition.

Proposition 5.4.5. *Let m be a natural number, and let $f : \mathbb{D}^{m+1} \rightarrow \mathbb{D}^{m+1}$ be a continuous map such that $f(\mathbb{S}^{m-1}) \subset \mathbb{S}^{m-1}$. If $f|_{\mathbb{S}^{m-1}} : \mathbb{S}^{m-1} \rightarrow \mathbb{S}^{m-1}$ is a homeomorphism, then f is surjective.*

Proof. Suppose that f is not surjective. Because \mathbb{D}^{m+1} is compact, the complement of the image of f is open in \mathbb{D}^{m+1} and therefore intersects $\text{Int}_{\mathbb{R}^{m+1}}(\mathbb{D}^{m+1})$.

Let p be a point in $\text{Int}_{\mathbb{R}^{m+1}}(\mathbb{D}^{m+1}) \cap (\text{Im } f)^c$, and recall the map $r_p : \mathbb{D}^{m+1} - \{p\} \rightarrow \mathbb{S}^{m-1}$,

$$x \in \mathbb{D}^{m+1} - \{p\} \mapsto \frac{1}{\|x - p\|}(x - p)$$

of Lemma 3.8.26. That same lemma proved that $r_p|_{\mathbb{S}^{m-1}} : \mathbb{S}^{m-1} \rightarrow \mathbb{S}^{m-1}$ is a homeomorphism.

By hypothesis, the composition $r_p \circ (f|_{\mathbb{S}^{m-1}}) : \mathbb{S}^{m-1} \rightarrow \mathbb{S}^{m-1}$ is a homeomorphism. Let $w : \mathbb{S}^{m-1} \rightarrow \mathbb{S}^{m-1}$ denote the inverse of $r_p \circ (f|_{\mathbb{S}^{m-1}})$. Then, $w \circ r_p \circ f : \mathbb{D}^{m+1} \rightarrow \mathbb{S}^{m-1}$ is a retraction. By contradiction with the nonexistence of such retractions (Theorem 3.8.25), f must be surjective. \square

6

An Application of the Roundness Theorem

6.1 INTRODUCTION

This chapter revisits the original impetus for the Roundness Theorem: the optimal control of quadratically-controlled control systems. The sole aim of the chapter is to disprove a conjecture that was made in the paper [40]. The conjecture appeared as Conjecture 1 in that paper, and is reproduced below as Conjecture 6.3.4. For brevity, the conjecture will henceforth be referred to as the KS Conjecture, for the authors (Khaneja and Stefanatos) who proposed it.

The roots of the KS-conjecture lie in the optimal control of a class of quadratically-controlled control systems that stems from system (1.2.1). However, the conjecture itself is stated in the terms of semidefinite

programming; specifically, the conjecture asserts that a certain class of semidefinite programs always has a rank 1 solution. Fortunately, the existence of rank 1 solutions to semidefinite programs is a problem that can be readily translated into the language of quadratic maps. Once translated, a judicious application of the Roundness Theorem will disprove the conjecture.

This chapter begins with an exploration of the optimal control problem that led to the KS Conjecture. Section 6.2 culminates in the statement of an alternative version of the KS conjecture; it, unlike the actual KS conjecture, is stated in terms of quadratic maps and is therefore more amenable to the Roundness Theorem. Section 6.3 makes the connection between the original KS conjecture and the version of it that is presented in Section 6.2. In the course of doing so, Section 6.3 discusses some very elementary aspects of semidefinite programming. Sections 6.4 and 6.5 carry out the refutation of the KS conjecture. Section 6.4 proves a proposition, Proposition 6.4.5, that is used in Section 6.5 to construct a counterexample to the conjecture.

6.2 THE KS CONJECTURE AND QUADRATIC MAPS

This section states a version of the KS conjecture; the version lends itself to refutation more easily than does the original conjecture. The equivalence between the two versions of the conjecture is discussed in Section 6.3.

The path to the KS conjecture begins with system (1.2.1), which is reproduced here for convenience,

$$\dot{x}(t) = \begin{pmatrix} \xi & -1 \\ 1 & \xi \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \bullet \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad (6.2.1)$$

Equation (6.2.1) makes a notational departure from (1.2.1); for brevity, (6.2.1) introduces the *Hadamard product*, $\bullet : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, which is defined with

respect to the standard coordinate system as the component-wise product,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \bullet \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 y_1 \\ \vdots \\ x_n y_n \end{pmatrix}$$

System (6.2.1) represents the dynamics by which coherence is transferred from one nuclear spin to another. The authors of [40] sought to determine the maximum efficiency with which such a transfer could take place. Their endeavour led them to pose the following problem,

Problem 6.2.1. *Let $\beta \in \mathbb{R}$, and let $\mathcal{R}_o \subset \mathbb{R}^2$ denote the set of points reachable from the origin $o \in \mathbb{R}^2$ along system (6.2.1).*

$$\begin{array}{ll} \text{Find} & \inf \{ \langle e_2, x \rangle_{\mathbb{R}^2} \mid x \in \mathcal{R}_o \} \\ \text{Subject to} & \langle e_1, x \rangle_{\mathbb{R}^2} = \beta \end{array} \quad (6.2.2)$$

(The heavy-handed notation $\langle e_i, x \rangle_{\mathbb{R}^2}$, as opposed to x_i , will justify itself later on when quadratic maps are brought to bear on Problem 6.2.1).

Let A denote the 2×2 matrix in (6.2.1),

$$A = \begin{pmatrix} \xi & -1 \\ 1 & \xi \end{pmatrix} \quad (6.2.3)$$

And, let $Q_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the quadratic map that appears on the right-hand side of (6.2.1), $Q_A(x) = Ax \bullet x$. Then, as was explained in Chapter 1, the reachable set \mathcal{R}_o is just $\text{Cl}_{\mathbb{R}^2}(\text{Conv}(\text{Im } Q_A))$, the closure of the convexification of $\text{Im } Q_A$. Moreover, since $\text{Conv}(\text{Im } Q_A)$ is dense in \mathcal{R}_o , and because $x \mapsto \langle e_2, x \rangle_{\mathbb{R}^2}$ is continuous, the infimum in Problem 6.2.1 need only be taken over $\text{Conv}(\text{Im } Q_A)$. Problem 6.2.1 can therefore be rephrased as follows,

Problem 6.2.2. Let $\beta \in \mathbb{R}$.

$$\begin{aligned} \text{Find} \quad & \inf \{ \langle e_2, x \rangle_{\mathbb{R}^2} \mid x \in \text{Conv}(\text{Im } Q_A) \} \\ \text{Subject to} \quad & \langle e_1, x \rangle_{\mathbb{R}^2} = \beta \end{aligned}$$

Remark 6.2.3. In Problem 6.2.2, $\text{Conv}(\text{Im } Q_A)$ took the place that $\mathcal{R}_o = \text{Cl}_{\mathbb{R}^2}(\text{Conv}(\text{Im } Q_A))$ had in Problem 6.2.1. This was done partly because it has no affect on the infimum in (6.2.2), but also because the KS conjecture deals with quadratic maps Q_A that are *definite*, and for which $\text{Conv}(\text{Im } Q_A)$ is actually equal to \mathcal{R}_o . ■

Intent on studying spin systems involving more than two spins, the authors of [40] ultimately posed the following higher-dimensional generalization of Problem 6.2.2,

Problem 6.2.4. Let $\beta \in \mathbb{R}^{n-1}$ (where \mathbb{R}^{n-1} is thought of as $e_n^\perp \subset \mathbb{R}^n$), and let A denote a real, $n \times n$ matrix. Let $Q_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the quadratic map $Q_A(x) = Ax \bullet x$.

$$\begin{aligned} \text{Find} \quad & \inf \{ \langle e_n, x \rangle_{\mathbb{R}^n} \mid x \in \text{Conv}(\text{Im } Q_A) \} \\ \text{Subject to} \quad & \pi_n(x) = \beta \end{aligned} \tag{6.2.4}$$

where $\pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the orthogonal projection onto $\mathbb{R}^{n-1} = e_n^\perp$.

It will be advantageous to make one further abstraction and state Problem 6.2.4 in terms of a general quadratic map.

Problem 6.2.5. Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map and let $\beta \in \mathbb{R}^{m-1} = e_m^\perp \subset \mathbb{R}^m$.

$$\begin{aligned} \text{Find} \quad & \inf \{ \langle e_m, x \rangle_{\mathbb{R}^m} \mid x \in \text{Conv}(\text{Im } Q) \} \\ \text{Subject to} \quad & \pi_m(x) = \beta \end{aligned} \tag{6.2.5}$$

where $\pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the orthogonal projection onto $\mathbb{R}^{n-1} = e_n^\perp$.

Let l_β be the affine line in \mathbb{R}^m that passes through the point $\beta \in \mathbb{R}^{m-1}$ and is parallel to e_m ,

$$l_\beta = \{\beta + te_m : t \in \mathbb{R}\} \quad (6.2.6)$$

Roughly speaking, Problem 6.2.5 asks how far one may descend down the line l_β (in the direction of $-e_m$), while remaining in $\text{Conv}(\text{Im } Q)$. Equivalently, under the identification of l_β with \mathbb{R} implied by (6.2.6), Problem 6.2.5 asks for $\inf l_\beta \cap \text{Im } Q$.

Definition 6.2.6. Let $n \in \mathbb{Z}^+$. Let $\beta \in \mathbb{R}^{m-1} = e_m^\perp \subset \mathbb{R}^m$, and let $Q \in \mathcal{Q}_{n,m}$ be a quadratic map. The pair (Q, β) is an **instance** of Problem 6.2.5.

Each instance of Problem 6.2.5 can be summarized broadly by three attributes.

Definition 6.2.7. An instance (Q, β) of Problem 6.2.5 is **feasible** if there exists an $x \in \text{Conv}(\text{Im } Q)$ satisfying $\pi_m(x) = \beta$. Equivalently, an instance of Problem 6.2.5 is feasible if $l_\beta \cap \text{Conv}(\text{Im } Q) \neq \emptyset$.

Definition 6.2.8. An instance (Q, β) of Problem 6.2.5 is **finite** if it is feasible and if the infimum in (6.2.5) is finite.

Definition 6.2.9. An instance (Q, β) of Problem 6.2.5 is **achievable** if it is finite and if there exists an $x_o \in \text{Conv}(\text{Im } Q)$ such that

$$\langle e_m, x_o \rangle_{\mathbb{R}^m} = \inf \{ \langle e_m, x \rangle_{\mathbb{R}^m} \mid x \in \text{Conv}(\text{Im } Q), \pi_m(x) = \beta \}$$

In this case, the instance (Q, β) is said to be **achieved** by x . Equivalently, (Q, β) is achievable if, under the identification of l_β with \mathbb{R} implied by (6.2.6), $l_\beta \cap \text{Conv}(\text{Im } Q)$ is a left-closed interval.

Remark 6.2.10. Definition 6.2.9 has a quirk: if $\text{Conv}(\text{Im } Q) \subsetneq \mathcal{R}_o = \text{Cl}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$, then there may exist an instance (Q, β) that is not achievable, but for which there exists an $x \in \mathcal{R}_o - \text{Conv}(\text{Im } Q)$ such that $\langle e_m, x \rangle_{\mathbb{R}^m}$ is equal to the infimum in (6.2.5). This quirk will not affect this

chapter's progress; all of the quadratic maps Q that are of interest to the KS conjecture are *definite*, and therefore satisfy $\text{Conv}(\text{Im } Q) = \mathcal{R}_o$. ■

The KS conjecture is about a fourth attribute of instances of 6.2.5, a distinguished kind of achievability. Within the reachable set \mathcal{R}_o of system (6.2.1) is the set of points that can be reached from the origin using a constant control. Section 1.2 showed that this set is simply $\text{Im } Q_A$. The next definition singles out those instances of Problem 6.2.5 that can be achieved by points that are in the image of Q .

Defintion 6.2.11. *An instance (Q, β) of Problem 6.2.5 is **simply-achievable** if it is achievable and if there exists an $x_o \in \text{Im } Q$ such that*

$$\langle e_m, x_o \rangle_{\mathbb{R}^m} = \inf \{ \langle e_m, x \rangle_{\mathbb{R}^m} \mid x \in \text{Conv}(\text{Im } Q), \pi_m(x) = \beta \}$$

Some of the attributes just listed are tautologically implied by some of the others. For example, if an instance of Problem 6.2.5 is finite, it must be feasible. To be complete,

$$\text{simply-achievable} \Rightarrow \text{achievable} \Rightarrow \text{finite} \Rightarrow \text{feasible} \quad (6.2.7)$$

No other implications are true. As this chapter unfolds it will present examples of feasible instances of Problem 6.2.5 that are not finite, finite instances that are not achievable, and achievable instances that are not simply-achievable.

When $\xi > 0$ in the matrix A of (6.2.3), A has a curious property: every feasible instance (Q_A, β) of Problem 6.2.4 that involves Q_A is simply-achievable. Each converse in the chain of implications (6.2.7) holds true for every instance (Q_A, β) . This property of A owes itself to a single characteristic, namely that when $\xi > 0$, the symmetric part of A is positive definite: $A + A^T \in \mathring{S}_n^+$. Section 6.5 will explain in detail how the positive definiteness of $A + A^T$ grants A the foregoing property. For now, a sketch of the argument, buttressed by some as yet unproven assertions, will have to suffice. The linchpin of the argument is a pair of facts.

First, the positive definiteness of $A + A^T$ (when $\xi > 0$) implies that $\text{Im } Q_A$ is a

definite cone containing the positive orthant \mathbb{R}_+^2 of \mathbb{R}^2 . In addition, by Dine's theorem (Theorem 2.1.1), $\text{Im } Q_A$ is convex. Thus, when $\xi > 0$, the image of Q_A looks something like the cone depicted in Figure 6.2.1.

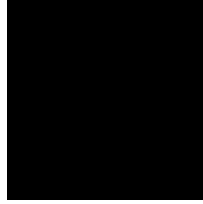


Figure 6.2.1: The image of Q_A when $\xi > 0$.

A pictorial argument can now demonstrate that when $\xi > 0$, every feasible instance (A, β) involving A is indeed simply-achievable; see Figure 6.2.2.

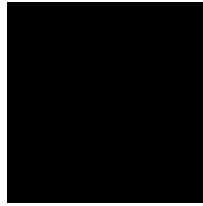


Figure 6.2.2: The intersection of l_β and $\text{Conv}(\text{Im } Q_A)$ (which is equal to $\text{Im } Q_A$). Recall that l_β is the line described in (6.2.6).

For any $\beta \in \mathbb{R}^1 = e_2^\perp \subset \mathbb{R}^2$ such that $l_\beta \cap \text{Im } Q_A \neq \emptyset$, there is a point $x \in \text{Im } Q_A$ for which $\langle e_2, x \rangle_{\mathbb{R}^2} = \inf l_\beta \cap \text{Im } Q_A$. It follows that (Q_A, β) is simply-achievable.

A question presents itself at this juncture: if A is *any* $n \times n$ matrix, does the positive definiteness of $A + A^T$ alone guarantee that every feasible instance of Problem 6.2.5 involving Q_A is simply-achievable? In an attempt to address this question, the authors of [40] numerically solved a large number of instances of Problem 6.2.5, for varying n , β and A , all satisfying $A + A^T \in \mathring{S}_n^+$ (Problem 6.2.5 can be numerically solved by casting it as a *semidefinite program*, for which many software packages exist; this is discussed in Section 6.3). The authors of [40]

found that each instance of Problem 6.2.5 they solved was simply-achievable; they conjectured that this is *always* the case.

Conjecture 6.2.12. *Let $n \in \mathbb{Z}^+$. Let A be an $n \times n$ matrix satisfying $A + A^T \in \mathring{S}_n^+$ and let $\beta \in \mathbb{R}^{n-1} = e_n^\perp \subset \mathbb{R}^n$. If (Q_A, β) is a feasible instance of Problem 6.2.5, then it is simply-achievable.*

The version of the KS conjecture that this section sought is not Conjecture 6.2.12, but rather a corollary of it.

Conjecture 6.2.13 (The KS Conjecture, Version I). *Let $n \in \mathbb{Z}^+$. Let A be an $n \times n$ matrix satisfying $A + A^T \in \mathring{S}_n^+$. If (Q_A, β) is feasible for all $\beta \in \mathbb{R}^{n-1} = e_n^\perp = \mathbb{R}^n$, then (Q_A, β) is simply-achievable for all $\beta \in \mathbb{R}^{n-1}$.*

It will be useful to have a vocabulary for describing conditions that are true for all $\beta \in \mathbb{R}^{m-1}$.

Definition 6.2.14. *A quadratic map $Q \in \mathcal{Q}_{n,m}$ is **completely feasible** if every instance of Problem 6.2.5 involving Q is feasible, **completely finite** if every instance of Problem 6.2.5 involving Q is finite, **completely achievable** if every instance of Problem 6.2.5 involving Q is achievable, and **completely simply-achievable** if every instance of Problem 6.2.5 involving Q is simply-achievable.*

In the terms of Definition 6.2.14, Version I of the KS conjecture becomes this,

Conjecture 6.2.15 (The KS Conjecture, Version I). *Let $n \in \mathbb{Z}^+$. Let A be an $n \times n$ matrix satisfying $A + A^T \in \mathring{S}_n^+$. If Q_A is completely feasible, then it is completely simply-achievable.*

If an instance (Q, β) of Problem 6.2.5 is achievable, then the point x_0 that achieves it cannot lie in the interior of $\text{Conv}(\text{Im } Q)$; it must lie on the boundary $\partial_{\mathbb{R}^n} \text{Conv}(\text{Im } Q)$. If (Q, β) is simply achievable, then x_0 must lie on the boundary of $\text{Im } Q$ itself. The complete simple-achievability of Q , then, is closely related to its boundary-convexity (recall Definition 4.5.11). This will drive the search for a counterexample to the KS conjecture.

The remainder of this chapter begins with a digression. The KS conjecture was not originally stated in the terms of Conjecture 6.2.15; in its original form it was a statement about a certain class of semidefinite programs. Section 6.3 establishes a connection between Conjecture 6.2.15 and the true KS conjecture. Section 6.4 establishes a necessary and sufficient condition for a matrix A to be completely simply-achievable (see Proposition 6.4.5). Finally, Section 6.5 uses Theorem 6.4.5 to construct a counterexample to Conjecture 6.2.15: a completely feasible matrix A satisfying $A + A^T \in \mathring{S}_n^+$, but which is not completely simply-achievable.

6.3 SEMIDEFINITE PROGRAMS AND QUADRATIC MAPS

This section states the KS conjecture in its original form, the one in which it appears in [40]. In that form, the KS conjecture was a statement about a certain class of *semidefinite programs*.

Defnition 6.3.1. Let $n, m \in \mathbb{Z}^+$. Let $A_i \in \text{Sym}_n$ for $i = 1, \dots, m$ and let $\beta = (\beta_1, \dots, \beta_{m-1}) \in \mathbb{R}^{m-1}$. The following optimization problem is the **semidefinite program** (or **SDP**) based on A_1, \dots, A_m and β ,

$$\begin{array}{ll} \text{Find} & \inf \{ \langle M, A_m \rangle_{\text{Sym}_n} \} \\ \text{Subject to} & M \in S_n^+ \end{array}$$

(The inner product $\langle \cdot, \cdot \rangle_{\text{Sym}_n}$ is taken to be the trace inner product on Sym_n induced by the standard inner product on \mathbb{R}^n).

The conditions $\langle M, A_i \rangle_{\text{Sym}_n} = \beta_i$ define an affine subspace $P \subset \text{Sym}_n$. The semidefinite program based on A_1, \dots, A_m and β asks for the minimum value (actually, the infimum of the values) taken by the linear function $M \mapsto \langle M, A_m \rangle_{\text{Sym}_n}$ on the convex subset $P \cap S_n^+$.

Like Problem 6.2.4, semidefinite programs can be feasible, finite or achievable,

Defnition 6.3.2. Let n, m, β and A_i be defined as they were in Defnition 6.3.1. The SDP based on A_1, \dots, A_m and β is **feasible** if there exists an $M \in S_n^+$ satisfying

$\langle M, A_i \rangle_{\text{Sym}_n} = \beta_i$ for $i = 1, \dots, m-1$; such an M is itself **feasible**. The SDP is **finite** if it is feasible and if the infimum in Definition 6.3.1 is finite. The SDP is **achievable** if it is finite and if there exists a feasible $M \in S_n^+$ for which $\langle A_m, M \rangle_{\text{Sym}_n}$ is equal to the infimum in Definition 6.3.1; such an M is a **solution** to the SDP.

Definition 6.3.3. A semidefinite program is **k-solvable** if it has a solution, $M \in \text{Sym}_n$, and if $\text{rank } M = k$.

Suppose that the semidefinite program based on $A_1, \dots, A_m \in \text{Sym}_n$ and $\beta \in \mathbb{R}^{m-1}$ is finite, and let α denote the infimum of Definition 6.3.1. Let $P \subset \text{Sym}_n$ again denote the affine subspace of Sym_n consisting of those $M \in \text{Sym}_n$ satisfying $\langle M, A_i \rangle_{\text{Sym}_n} = \beta_i$ and let $H_{A_m}^\alpha$ denote the following affine hyperplane,

$$H_{A_m}^\alpha = \{M \in \text{Sym}_n \mid \langle M, A_m \rangle_{\text{Sym}_n} = \alpha\}$$

To say that the SDP based on A_1, \dots, A_m and β is solvable is to say that $H_{A_m}^\alpha \cap P \cap S_n^+$ is nonempty. To say that SDP is k -solvable is to say, in the notation of Section 3.8.3, that $H_{A_m}^\alpha \cap P \cap S_{n,k}^+$ is nonempty, where $S_{n,k}^+$ is the set of positive semidefinite elements of Sym_n having rank k .

To state the original KS conjecture, let A be an $n \times n$ matrix, and, for $i = 1, \dots, n$, let $A_i = E_i A + A^T E_i$, where E_i is the $n \times n$ matrix having a 1 in its i^{th} diagonal entry, and zeros everywhere else. The original KS conjecture was this,

Conjecture 6.3.4 (The KS Conjecture). *Let A and A_i be defined as they were above. Let $\beta = (\beta_1, \dots, \beta_{m-1}) \in \mathbb{R}^{m-1}$. If the SDP based on A_1, \dots, A_m and β is feasible, then it is 1-solvable.*

The remainder of this section establishes a correspondence between the original KS conjecture and the preliminary Conjecture 6.2.12. To begin, consider the following assertions,

- Each instance of Problem 6.2.4 defines a semidefinite program.

- An instance of Problem 6.2.4 is feasible, finite or achievable if and only if its counterpart semidefinite program is the same.
- An instance of Problem 6.2.4 is simply-achievable if and only if its counterpart semidefinite program is 1-solvable.

Granted these assertions, the original KS conjecture is true if and only if Conjecture 6.2.12 is true. That the assertions themselves are true follows from a simple calculation. Let $Q \in \mathcal{Q}_{n,m}$ be any quadratic map and let $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ be its ALM (Q will soon be specialized to the map $x \mapsto Ax \bullet x$). Every point in $\text{Conv}(\text{Im } Q)$ is of the form

$$x = \sum_{i=1}^k Q(x_i)$$

for some $x_i \in \mathbb{R}^n$. Thus, for any $j = 1, \dots, m$,

$$\begin{aligned} \langle x, e_j \rangle_{\mathbb{R}^m} &= \sum_{i=1}^k \langle Q(x_i), e_j \rangle_{\mathbb{R}^m} \\ &= \sum_{i=1}^k \langle x_i, \psi(e_j)x_i \rangle_{\mathbb{R}^n} \\ &= \sum_{i=1}^k \langle \psi(e_j), x_i \otimes x_i \rangle_{\text{Sym}_n} \\ &= \langle \psi(e_j), M \rangle_{\text{Sym}_n} \end{aligned} \tag{6.3.1}$$

where $M = \sum_{i=1}^k x_i \otimes x_i \in S_n^+$.

Let A be an $n \times n$ matrix and let ψ_A denote the ALM of Q_A . It follows from (6.3.1) that, for $\beta \in \mathbb{R}^{n-1}$, the correspondence that associates the instance (Q_A, β) of Problem 6.2.4 with the SDP defined by $\psi_A(e_1), \dots, \psi_A(e_n)$ and β affirms the three assertions above. The two versions of the KS conjecture given above are equivalent.

6.4 COMPLETE SIMPLE ACHIEVABILITY

A quadratic map $Q \in \mathcal{Q}_{n,m}$ cannot be completely simply achievable unless it is completely feasible, completely finite and completely achievable. This section presents necessary and sufficient conditions for each of these properties and culminates in a necessary and sufficient condition for complete simple achievability.

It will be assumed throughout this section that Q is a definite quadratic map. This is done partly for ease, but mostly because Q_A is definite when $A + A^T \in \mathring{S}_n^+$ (this will be shown ahead).

6.4.1 COMPLETE FEASIBILITY

A quadratic map $Q \in \mathcal{Q}_{n,m}$ is completely feasible if and only if every affine line in \mathbb{R}^m that is parallel to e_m (that is, every affine line of the form l_β from (6.2.6)) intersects $\text{Conv}(\text{Im } Q)$. Equivalently, the projection of $\text{Conv}(\text{Im } Q)$ onto $e_m^\perp = \mathbb{R}^{m-1}$ must be surjective. In particular, one of either e_m or $-e_m$ must be contained in $\text{Conv}(\text{Im } Q)$.

Neither of the memberships $\pm e_m \in \text{Conv}(\text{Im } Q)$ is a sufficient condition for complete feasibility. For example, e_m is contained in the image of the standard quadratic map but the standard quadratic map is not completely feasible. However, it does suffice for either e_m or $-e_m$ to be contained in the *interior* of $\text{Conv}(\text{Im } Q)$.

Proposition 6.4.1. *Let $Q \in \mathcal{Q}_{n,m}$. Q is completely feasible if and only if $\langle e_m \rangle \cap \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q)) \neq \emptyset$.*

Proof. Let $K \subset \mathbb{R}^m$ be the cone $\pi_m(\text{Conv}(\text{Im } Q))$, where $\pi_m : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the orthogonal projection onto e_m^\perp . Then,

Q is completely feasible

$$\Leftrightarrow K = e_m^\perp$$

$$\Leftrightarrow K^* = \langle e_m \rangle$$

On the other hand, by Proposition 3.4.4.5, $K^* = \pi_m^{-1}(\text{Conv}(\text{Im } Q)^*)$. Thus,

Q is completely feasible

$$\Leftrightarrow \pi_m^{-1}(\text{Conv}(\text{Im } Q)^*) = \langle e_m \rangle$$

$$\Leftrightarrow \text{Conv}(\text{Im } Q)^* \cap e_m^\perp = \{0\}$$

Given that Q is definite, $\text{Conv}(\text{Im } Q)^* \cap e_m^\perp = \{0\}$ if and only if $\psi(e_m^\perp)$ is an indefinite subspace of Sym_n , where $\psi \in \mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ is the ALM of Q . By Proposition 4.4.8, $\psi(e_m^\perp)$ is indefinite if and only if $\langle e_m \rangle \cap \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q)) \neq \emptyset$. □

6.4.2 COMPLETE FINITENESS

Suppose that Q is completely feasible. Then, for each $\beta \in e_m^\perp = \mathbb{R}^{m-1}$, $\beta + te_m$ is in $\text{Conv}(\text{Im } Q)$ for some $t \in \mathbb{R}$. In addition, by Proposition 6.4.1, one or both of e_m and $-e_m$ is contained in $\text{Conv}(\text{Im } Q)$. If $-e_m \in \text{Conv}(\text{Im } Q)$, then, for all $\beta \in \mathbb{R}^{m-1}$, $\beta - te_m$ is contained in $\text{Conv}(\text{Im } Q)$ for all sufficiently large, nonnegative t . It follows that if $-e_m \in \text{Conv}(\text{Im } Q)$, not a single instance of Problem 6.2.5 is finite. When Q is definite, the converse is true as well.

Proposition 6.4.2. *Let $Q \in \mathcal{Q}_{n,m}$ be a definite quadratic map. Then, Q is completely finite if and only if it is completely feasible and $-e_m \notin \text{Conv}(\text{Im } Q)$. Equivalently, by Proposition 6.4.1, Q is completely finite if and only if $e_m \in \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$.*

Proof. The preamble to the proposition showed that Q cannot be completely finite if $-e_m \in \text{Conv}(\text{Im } Q)$. For the converse statement, suppose that (Q, β) is a feasible, infinite instance of Problem 6.2.5. Then, $\beta - te_m$ is contained in $\text{Conv}(\text{Im } Q)$ for all sufficiently large, $t \in \mathbb{R}$. But then,

$$\frac{1}{\|\beta - te_m\|} (\beta - te_m) \tag{6.4.1}$$

is in $\text{Conv}(\text{Im } Q)$ as well (for all sufficiently large t). The element (6.4.1)

approaches $-e_m$ as $t \rightarrow \infty$. The closedness of $\text{Conv}(\text{Im } Q)$ (itself a consequence of the definiteness of Q) then implies that $-e_m \in \text{Conv}(\text{Im } Q)$. \square

6.4.3 COMPLETE ACHIEVABILITY

If Q is definite, the condition of Proposition 6.4.2 is also equivalent to complete achievability.

Proposition 6.4.3. *Let $Q \in \mathcal{Q}_{n,m}$ be a definite quadratic map. Then, Q is completely achievable if and only if $e_m \in \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$.*

Proof. Let $\beta \in \mathbb{R}^{m-1} = e_m^\perp \subset \mathbb{R}^m$. The closedness of $\text{Conv}(\text{Im } Q)$ and the complete finiteness of Q imply that $l_\beta \cap \text{Conv}(\text{Im } Q)$ is a left-closed interval (under the identification of l_β with \mathbb{R} implied by (6.2.6)). It follows that (Q, β) is achievable. \square

6.4.4 COMPLETE SIMPLE-ACHIEVABILITY

Suppose that $Q \in \mathcal{Q}_{n,m}$ (still definite) is completely achievable, and let $x \in \text{Conv}(\text{Im } Q)$ be a solution to a given instance (Q, β) of Problem 6.2.5 (note then that $\pi_m(x) = \beta$). The point x then necessarily lies on the boundary of $\text{Conv}(\text{Im } Q)$.

If, conversely, x is a point lying on the boundary of $\text{Conv}(\text{Im } Q)$, then the following proposition shows that x is the unique solution to the instance $(Q, \pi_m(x))$.

Proposition 6.4.4. *Let $Q_{n,m}$ be definite and completely achievable. Let $x \in \partial_{\mathbb{R}^m} \text{Conv}(\text{Im } Q)$. Then, for all $t > 0$, $x - te_m \notin \text{Conv}(\text{Im } Q)$ and $x + te_m \in \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$ for all $t > 0$.*

Proof. By Proposition 6.4.3, $e_m \in \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$. Thus, $\langle u, e_m \rangle > 0$ for all nonzero $u \in \text{Conv}(\text{Im } Q)^*$. Also, because x lies on the boundary of $\text{Conv}(\text{Im } Q)$, it lies in $u^\perp \cap \text{Conv}(\text{Im } Q)$ for some nonzero $u_x \in \text{Conv}(\text{Im } Q)^*$. The two assertions in the proposition now follow. First, it follows that $\langle x - te_m, u_x \rangle < 0$

for all $t > 0$, and therefore that $x - te_m \notin \text{Conv}(\text{Im } Q)$. Second, for $t > 0$, $\langle x + te_m, u \rangle > 0$ for all nonzero $u \in \text{Conv}(\text{Im } Q)^*$, and therefore $x + te_m \in \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$. \square

If Q is completely simply-achievable, it must therefore be the case that every $x \in \partial_{\mathbb{R}^m} \text{Conv}(\text{Im } Q)$ is contained in $\text{Im } Q$ itself. Q , in other words, is boundary-convex (see Definition 4.5.11).

Proposition 6.4.5. *Let $Q \in \mathcal{Q}_{n,m}$ be a definite quadratic map. Q is completely simply-achievable if and only if it is boundary-convex and $e_m \in \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q))$.*

Proposition 6.4.5 invites the Roundness Theorem to the search for a counterexample to the KS conjecture. If an $n \times n$ matrix A is such that the quadratic map Q_A is not round, then Q_A is likely not boundary convex. If Q_A is definite and satisfies $e_m \in \text{Int}_{\mathbb{R}^m}(\text{Conv}(\text{Im } Q_A))$, then Q_A would stand as a potential counterexample to the KS conjecture. The search for such a matrix, and the confirmation that Q_A is not only nonround but not boundary-convex as well, will occupy the remainder of this chapter.

6.4.5 AN ADDENDUM

Section 6.2 promised examples of feasible instances of Problem 6.2.5 that are not finite, finite instances that are not achievable, and achievable instances that are not simply-achievable. The first two of these examples are presented here. The counterexample to the KS conjecture that is presented in Section 6.5 provides the third example.

Example 6.4.6. *Let $Q \in \mathcal{Q}_n$ be the negative of the standard quadratic map. Then, if all of the components of $\beta \in \mathbb{R}^{n-1}$ are nonpositive, (Q, β) is a feasible, infinite instance of Problem 6.2.5.*

Example 6.4.7. *Let $Q \in \mathcal{Q}_2$ be the following quadratic map,*

$$Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1x_2 \\ x_1^2 \end{pmatrix}$$

Then,

$$\text{Im } Q = \{0\} \cup \mathring{H}_{e_2}^+$$

(Recall that $\mathring{H}_{e_2}^+$ is the open upper-half space $\{x \in \mathbb{R}^2 \mid \langle x, e_2 \rangle_{\mathbb{R}^2} > 0\}$). It follows that Q is completely finite. It also follows that no instance of Problem 6.2.5 that involves Q is achievable.

6.5 A COUNTEREXAMPLE

This section constructs a counterexample to the KS conjecture (more precisely, to Conjecture 6.2.15). The counterexample consists of a 4×4 matrix A satisfying $A + A^T \in \mathring{S}_4^+$, and for which Q_A is completely feasible, but not completely simply-achievable. An appeal to Proposition 6.4.1 will suffice to show that Q_A is completely feasible. Showing that Q_A is not completely simply-achievable will require more effort; it will require a search for a nonconvex face of $\text{Im } Q_A$. Both steps will require detailed knowledge of the ALM of Q_A . The first proposition in this section provides just that.

Definition 6.5.1. For $u \in \mathbb{R}^n$, $\mathbf{D}_u \in \text{Sym}_n$ will denote the diagonal matrix whose diagonal is u .

Remark 6.5.2. In terms of Definition 6.5.1, the Hadamard product is given by $x \bullet y = D_x y = D_y x$. Note that $\langle x \bullet y, z \rangle = \langle x, y \bullet z \rangle$. ■

Proposition 6.5.3. Let $n \in \mathbb{Z}^+$ and let A be an $n \times n$ matrix. Let $Q_A \in \mathcal{Q}_n$ denote the quadratic map $x \mapsto Ax \bullet x$. The ALM of Q_A is $\psi_A(u) = \frac{1}{2} (D_u A + A^T D_u)$.

Proof. By definition, ψ_A is the unique map in $\mathcal{L}(\mathbb{R}^n, \text{Sym}_n)$ satisfying

$$\langle x, \psi_A(u)x \rangle_{\mathbb{R}^n} = \langle Q_A(x), u \rangle_{\mathbb{R}^n}$$

So,

$$\begin{aligned}
\langle x, \psi_A(u)x \rangle_{\mathbb{R}^n} &= \langle Ax \bullet x, u \rangle_{\mathbb{R}^n} \\
&= \langle Ax, u \bullet x \rangle_{\mathbb{R}^n} \\
&= \langle Ax, D_u x \rangle_{\mathbb{R}^n} \\
&= \langle x, A^T D_u x \rangle_{\mathbb{R}^n} \\
&= \frac{1}{2} \langle x, (A^T D_u + D_u A^T) x \rangle_{\mathbb{R}^n}
\end{aligned}$$

□

Note that the entries of the matrix $\psi_A(u)$ are given by $(\psi_A(u))_{i,j} = a_{i,j}u_i + a_{j,i}u_j$.

Corollary 6.5.4. *Using the notation of Proposition 6.5.3, if $A + A^T \in \mathring{S}_n^+$, then Q_A is a definite quadratic map.*

Proof. The all-ones element $u = (1, \dots, 1) \in \mathbb{R}^n$ is mapped by ψ_A to $A + A^T$. □

Proposition 6.5.5. *The following matrix is a counterexample to the KS conjecture (Conjecture 6.2.15),*

$$A = 2 \cdot \begin{pmatrix} 1 & \varepsilon & 0 & 0 \\ -\varepsilon & 1 & \varepsilon & 0 \\ 0 & -\varepsilon & 1 & \varepsilon \\ 0 & 0 & -\varepsilon & 1 \end{pmatrix}$$

The remainder of this section consists of a proof of Proposition 6.5.5. The proof must show three things: that $A + A^T \in \mathring{S}_4^+$, that Q_A is completely feasible, and that Q_A is *not* completely simply-achievable. That $A + A^T \in \mathring{S}_4^+$ is clear. A couple of preliminary observations will prove useful in the two remaining steps.

First, the quadratic map Q_A and its ALM are respectively given by,

$$Q_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 2 \cdot \begin{pmatrix} x_1^2 + \varepsilon x_1 x_2 \\ x_2^2 - \varepsilon x_1 x_2 + \varepsilon x_2 x_3 \\ x_3^2 - \varepsilon x_2 x_3 + \varepsilon x_3 x_4 \\ x_4^2 - \varepsilon x_3 x_4 \end{pmatrix}$$

$$\psi_A(u) = \begin{pmatrix} 2u^1 & \varepsilon(u^1 - u^2) & 0 & 0 \\ \varepsilon(u^1 - u^2) & 2u^2 & \varepsilon(u^2 - u^3) & 0 \\ 0 & \varepsilon(u^2 - u^3) & 2u^3 & \varepsilon(u^3 - u^4) \\ 0 & 0 & \varepsilon(u^3 - u^4) & 2u^4 \end{pmatrix} \quad (6.5.1)$$

for $u = (u^1, u^2, u^3, u^4) \in \mathbb{R}^4$ (this uses Proposition 6.5.3). Second, the 2×2 block form of A is,

$$A = 2 \cdot \begin{pmatrix} \Theta & B \\ -B^T & \Theta \end{pmatrix} \quad (6.5.2)$$

where

$$B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix} \text{ and } \Theta = \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \quad (6.5.3)$$

The next proposition, together with Proposition 6.4.1 shows that Q_A is completely feasible.

Proposition 6.5.6. $e_4 \in \text{Int}_{\mathbb{R}^+}(\text{Conv}(\text{Im } Q_A))$.

Proof. Note that $e_4 \in \text{Im } Q_A$, since $Q_A(e_4) = e_4$. It remains to show that $e_4 \in \text{Int}_{\mathbb{R}^+}(\text{Conv}(\text{Im } Q_A))$. By Corollary 6.5.4, $\text{Conv}(\text{Im } Q_A)$ is a definite cone. Thus, e_4 is in $\text{Int}_{\mathbb{R}^+}(\text{Conv}(\text{Im } Q_A))$ if and only if $\psi_A(e_4^\perp)$ is an indefinite subspace of Sym_4 .

Now, $\psi_A(e_4^\perp)$ is given by

$$\psi_A(e_4^\perp) = \left\{ \begin{pmatrix} 2u^1 & \varepsilon(u^1 - u^2) & 0 & 0 \\ \varepsilon(u^1 - u^2) & 2u^2 & \varepsilon(u^2 - u^3) & 0 \\ 0 & \varepsilon(u^2 - u^3) & 2u^3 & \varepsilon u^3 \\ 0 & 0 & \varepsilon u^3 & 0 \end{pmatrix} : u^1, u^2, u^3 \in \mathbb{R} \right\} \quad (6.5.4)$$

Suppose that $\psi_A(u) \in \mathring{S}_4^+$ for some $u \in e_4^\perp$. The 0 in the $(4, 4)$ -entry of the matrix in (6.5.4) forces the equality $u^3 = 0$ (if a positive semidefinite matrix has a 0 in its i^{th} diagonal, its i^{th} row and i^{th} column must consist entirely of zeros). The resulting 0 in the $(3, 3)$ -entry of (6.5.4) similarly forces $u^2 = 0$. Finally, $u^2 = 0$ forces the second column of (6.5.4), and therefore u^1 to be zero as well. Thus, $u = 0$. It follows that $\psi_A(e_4^\perp)$ is an indefinite subspace of Sym_4 and therefore that $e_4 \in \text{Int}_{\mathbb{R}^4}(\text{Conv}(\text{Im } Q_A))$. \square

All that remains now is to show that Q_A is not completely simply-achievable. This, by Proposition 6.4.5, is equivalent to showing that $\text{Im } Q_A \subset \mathbb{R}^4$ has a nonconvex face. Section 5.3 in Chapter 5 prescribed the following procedure for showing that a quadratic map $Q \in \mathcal{Q}_n$ has a nonconvex face (as usual, ψ denotes the ALM of Q),

1. Show that Q is generically nonround.
2. Find a $u \in \mathbb{R}^n$ for which $\psi(u) \in S_n^+$ and $1 < \text{rank } \psi(u) < n - 1$ (u exists because Q is generically nonround).
3. Let $\bar{Q} : \ker \psi(u) \rightarrow u^\perp$ denote the quadratic map $\bar{Q} = \pi_{u^\perp} \circ Q|_{\ker \psi(u)}$, where π_{u^\perp} is the orthogonal projection from \mathbb{R}^n to u^\perp . Let $\bar{\psi} \in \mathcal{L}(u^\perp, \text{Sym}(\ker \psi(u)))$ denote the ALM of \bar{Q} . Q is not convex if $\bar{\psi}$ has full rank.

According to Section 5.3, Step 3 can be supplanted by showing that $\ker \psi' = \langle u \rangle$, where ψ' is the ALM of the restriction $Q|_{\ker \psi(u)}$. This eliminates the need to

compute the projection π_{u^\perp} .

The remainder of this section consists of the application of the preceding procedure to Q_A . This venture requires a brief digression on a particular quadratic map from \mathbb{R}^2 to itself.

A DIGRESSION

Let $P \in \mathcal{Q}_2$ be the quadratic map $P(x) = \Theta x \bullet x$, where Θ is the 2×2 matrix given in (6.5.3), and appearing in the block form of A .

The linear map $\psi_P \in \mathcal{L}(\mathbb{R}^2, \text{Sym}_2)$ associated with P is given by

$$\psi_P(u) = \begin{pmatrix} 2u^1 & \varepsilon(u^1 - u^2) \\ \varepsilon(u^1 - u^2) & 2u^2 \end{pmatrix} \quad (6.5.5)$$

for $u = (u^1, u^2) \in \mathbb{R}^2$. It follows that P is nondegenerate and definite (the latter follows by setting $u^1 = u^2 > 0$). The definiteness of P , or equivalently of ψ_P , implies that $K_{\psi_P} \neq \{0\}$ and therefore $\partial_{\mathbb{R}^2} K_{\psi_P} \neq \{0\}$. Thus, there exists a nonzero $z^* = (z^1, z^2) \in \mathbb{R}^2$ for which $\psi_P(z^*)$ is positive semidefinite with rank 1.

Finally, note that nonzero elements of \mathbb{R}^2 of the form $(u^1, 0)$ and $(0, u^2)$ are mapped by ψ_P to *indefinite* elements of Sym_2 . It follows that *both* of the components, z^1 and z^2 , of z^* are nonzero. The diagonal elements of $\psi_P(z^*)$ are $2z^1$ and $2z^2$; z^1 and z^2 are therefore positive.

Remark 6.5.7. If $v \in \ker \psi_P(z^*) \subset \mathbb{R}^2$, then $\langle \Theta v \bullet v, z^* \rangle_{\mathbb{R}^2} = 0$. It follows that $P(\ker \psi_P(z^*)) \subset \langle z^* \rangle^\perp$. Thus, if $w \in \mathbb{R}^2$ is an element satisfying $\langle \Theta v \bullet v, w \rangle_{\mathbb{R}^2} = 0$, then $w \in \langle z^* \rangle$.

Equation (6.5.5) and the positivity of z^1 and z^2 imply that the components of any nonzero $v \in \ker \psi_P(z^*)$ must both be nonzero. ■

Returning to the three step procedure for showing that Q_A has a nonconvex face, the following Proposition completes the Step 1.

Proposition 6.5.8. Q_A is generically nonround.

Proof. Let $\lambda = \frac{z^2}{z^1}$, where z^1 and z^2 are the components of the element $z^* \in \mathbb{R}^2$ found above. λ is positive. If $u_o = (z^1, z^2, z^2, \lambda z^1) = (z^*, \lambda z^*)$ then the $(2, 3)$ entry of $\psi_A(u_o)$ vanishes (see (6.5.1)), and

$$\psi_A(u_o) = \begin{pmatrix} \psi_P(z^*) & o_{2,2} \\ o_{2,2} & \lambda \psi_P(z^*) \end{pmatrix} \quad (6.5.6)$$

where $o_{2,2}$ denotes the 2×2 zero matrix. It follows from the definition of z^* that $\psi_A(u_o)$ is positive semidefinite and has rank 2. Q_A is generically nonround. \square

The element $u_o \in \mathbb{R}^4$ that was just constructed fulfills the requirements of Step 2 in the three-step procedure for showing that Q_A has a nonconvex face. All that remains now is Step 3.

Let $\psi'_A : \mathbb{R}^4 \rightarrow \text{Sym}(\ker \psi_A(u_o))$ denote the ALM of the restriction $Q_A|_{\ker \psi_A(u_o)}$. The task at hand is to show that $\ker \psi'_A = \langle u_o \rangle$. This task can be simplified by understanding the kernel $\ker \psi_A(u_o)$.

Claim 6.5.9. $\ker \psi(u_o) = \text{span } \{k_1, k_2\}$ where

$$k_1 = \begin{pmatrix} v^* \\ o_2 \end{pmatrix} \quad \text{and} \quad k_2 = \begin{pmatrix} o_2 \\ v^* \end{pmatrix}$$

where $v^* \in \mathbb{R}^2$ is any generating element of the 1-dimensional subspace $\ker \psi_P(z^*)$ and o_2 denotes $o \in \mathbb{R}^2$.

Proof. This follows from (6.5.6). \square

Fix a nonzero $v^* = (v^1, v^2) \in \ker \psi_P(z^*)$ and let $\{k_1, k_2\}$ be the corresponding basis provided by Claim 6.5.9. The basis $\{k_1, k_2\}$ defines an isomorphism $\varphi : \mathbb{R}^2 \rightarrow \ker \psi_A(u_o)$ given by $(x^1, x^2) \mapsto x^1 k_1 + x^2 k_2$.

The final task of this section is to show that the kernel of ψ'_A , the ALM of $Q_A|_{\ker \psi_A(u_o)}$ is equal to $\langle u_o \rangle$. It will be easier to work with the composition $Q_A|_{\ker \psi_A(u_o)} \circ \varphi$, rather than $Q_A|_{\ker \psi_A(u_o)}$ itself, simply because the composition $Q_A|_{\ker \psi_A(u_o)} \circ \varphi$ can be written in terms of the standard coordinates on \mathbb{R}^2 .

Proposition 4.6.2 says that the ALM of $Q_A|_{\ker \psi_A(u_o)} \circ \varphi$ is given by

$u \mapsto \varphi \cdot \psi'_A(u) \cdot \varphi^T$ where ψ'_A is the ALM of $Q_{\ker \psi_A(u_o)}$. It follows that the kernel of the ALM of $Q_A|_{\ker \psi_A(u_o)} \circ \varphi$ is equal to that of $Q_A|_{\ker \psi_A(u_o)}$.

It suffices, then, to calculate the ALM of $Q_A|_{\ker \psi_A(u_o)} \circ \varphi$. This can be done by first computing $Q_A|_{\ker \psi_A(u_o)}$ itself. This will require a few preliminary calculations.

First, using the block form (6.5.2),

$$Ak_1 = \begin{pmatrix} \Theta v^* \\ -B^T v^* \end{pmatrix} \quad \text{and} \quad Ak_2 = \begin{pmatrix} Bv^* \\ \Theta v^* \end{pmatrix}$$

It follows that

$$\begin{aligned} Ak_1 \bullet k_1 &= \begin{pmatrix} \Theta v^* \bullet v^* \\ \circ_2 \end{pmatrix} \\ Ak_2 \bullet k_1 &= \begin{pmatrix} \circ_2 \\ \Theta v^* \bullet v^* \end{pmatrix} \\ Ak_1 \bullet k_2 &= \begin{pmatrix} \circ \\ -B^T v^* \bullet v^* \end{pmatrix} \quad \text{and} \\ Ak_2 \bullet k_1 &= \begin{pmatrix} Bv^* \bullet v^* \\ \circ_2 \end{pmatrix} \end{aligned}$$

Note that

$$Bv^* \bullet v^* = \varepsilon \begin{pmatrix} \circ \\ v^1 v^2 \end{pmatrix} \quad \text{and} \quad -B^T v^* \bullet v^* = \varepsilon \begin{pmatrix} -v^1 v^2 \\ \circ \end{pmatrix}$$

The composition $Q|_{\ker \psi(u_o)} \circ \varphi$ can now be computed. For $x = (x^1, x^2) \in \mathbb{R}^2$,

$$\begin{aligned}
& (Q|_{\ker \psi(u_o)} \circ \varphi)(x) \\
&= (x^1 A k_1 + x^2 A k_2) \bullet (x^1 k_1 + x^2 k_2) \\
&= (x^1)^2 A k_1 \bullet k_1 + x^1 x^2 (A k_1 \bullet k_2 + A k_2 \bullet k_1) + (x^2)^2 A k_2 \bullet k_2 \\
&= (x^1)^2 \begin{pmatrix} \Theta v^* \bullet v^* \\ o_2 \end{pmatrix} + x^1 x^2 \begin{pmatrix} B v^* \bullet v^* \\ -B^T v^* \bullet v^* \end{pmatrix} + (x^2)^2 \begin{pmatrix} o_2 \\ \Theta v^* \bullet v^* \end{pmatrix}
\end{aligned} \tag{6.5.7}$$

$$\tag{6.5.8}$$

Finally, to compute the ALM of $Q|_{\ker \psi(u_o)} \circ \varphi$, let $u = (w, y) \in \mathbb{R}^4$, with $w = (w^1, w^2), y = (y^1, y^2) \in \mathbb{R}^2$. The ALM of $Q|_{\ker \psi(u_o)} \circ \varphi$ is determined by the quantity $\langle (Q|_{\ker \psi(u_o)} \circ \varphi)(x), u \rangle_{\mathbb{R}^4}$. Using (6.5.8) and the block-form $u = (w, y)$ of u ,

$$\begin{aligned}
& \langle (Q|_{\ker \psi(u_o)} \circ \varphi)(x), u \rangle_{\mathbb{R}^4} \\
&= (x^1)^2 \langle \Theta v^* \bullet v^*, w \rangle_{\mathbb{R}^2} + x^1 x^2 (\langle B v^* \bullet v^*, w \rangle_{\mathbb{R}^2} - \langle B^T v^* \bullet v^*, y \rangle_{\mathbb{R}^2}) \\
&\quad + (x^2)^2 \langle A v^* \bullet v^*, y \rangle_{\mathbb{R}^2}
\end{aligned} \tag{6.5.9}$$

Remark 6.5.10. Note that

$$\langle B v^* \bullet v^*, w \rangle_{\mathbb{R}^2} - \langle B^T k^* \bullet k^*, y \rangle_{\mathbb{R}^2} = v^1 v^2 (w^2 - y^1) \tag{6.5.10}$$

■

For u to be in the kernel of the ALM of $Q|_{\ker \psi(u_o)} \circ \varphi$, each of the coefficients in equation (6.5.9), thought of as a polynomial in x^1 and x^2 , must vanish. The coefficients of $(x^1)^2$ and $(x^2)^2$ are respectively $\langle \Theta v^* \bullet v^*, w \rangle$ and $\langle \Theta v^* \bullet v^*, y \rangle$. By Remark 6.5.7, the equalities $\langle \Theta v^* \bullet v^*, w \rangle = \langle \Theta v^* \bullet v^*, y \rangle = 0$ imply that w and y are both multiples of $z^* = (z^1, z^2)$.

Remark 6.5.7 also implies that $(v^*)^1 (v^*)^2 \neq 0$. Thus, the vanishing of the coefficient of $x^1 x^2$ in (6.5.9) implies that $w^2 = y^1$, by (6.5.10).

Remark 6.5.11. Recall that the components z^1 and z^2 of z^* are both nonzero. If

either of the components of a multiple of z^* vanishes, then that multiple must be 0.

By Remark 6.5.11, the equality $w^2 = y^1$ implies that either $w = y = 0$ or that w and y are both nonzero and therefore nonzero multiples of z^* . The equality $w^2 = y^1$ also implies that $y = \lambda w$ where $\lambda = \frac{z^2}{z^1}$ (this is the same λ that appeared in the proof of Proposition 6.5.8). Thus, if $w = az^*$ for some nonzero $a \in \mathbb{R}$, then $y = a\lambda z^*$. It follows that $u = a(z^*, \lambda z^*)$. But, u_o was defined in the proof of Proposition 6.5.8 as $(z^*, \lambda z^*)$. u is therefore in $\langle u_o \rangle$, and it follows that $\ker \psi'_A = \langle u_o \rangle$. It finally follows that Q_A has a nonconvex face.

7

A Miscellany of Minor Facts about Quadratic Maps

7.1 A NONCONVEX, BOUNDARY-CONVEX QUADRATIC MAP

This section presents a boundary-convex quadratic map $Q_H \in \mathcal{Q}_4$, a map satisfying $\partial_{\mathbb{R}^4} \text{Conv}(\text{Im } Q_H) \subset \text{Im } Q_H$, that is nevertheless nonconvex. The construction of the map depends heavily on the *Hopf map* [32], a well-known map from \mathbb{S}^3 to \mathbb{S}^2 that happens to be a quadratic map.

Defintion 7.1.1. The *extended Hopf map* is the quadratic map $\overline{Q}_H : \mathbb{R}^4 \rightarrow \mathbb{R}^3$

defined by the matrices

$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The **Hopf map** is the restriction $\overline{Q}_H|_{\mathbb{S}^3}$. Finally, the **Conic Hopf map** is the quadratic map $Q_H \in \mathcal{Q}_4$ given by

$$x \in \mathbb{R}^4 \mapsto \begin{pmatrix} \overline{Q}_H(x) \\ ||x||^2 \end{pmatrix}$$

Proposition 7.1.2. *The conic Hopf map Q_H is boundary-convex (that is, it satisfies $\partial_{\mathbb{R}^4} \text{Conv}(\text{Im } Q_H) \subset \text{Im } Q_H$), but is not convex.*

Remark 7.1.3. Before proving the proposition, it is important to note that the conic Hopf map does not contradict the Roundness Theorem, for it is not round. To see this, note that, in the notation of Definition 7.1.1, $M_3 + I_4$ is in ∂S_4^+ , but has rank 2.

The proof of this rests on a single property of the Hopf map, $\overline{Q}_H|_{\mathbb{S}^3}$, namely that its image is equal to $\mathbb{S}^2 \subset \mathbb{R}^3$ [32].

Proof. (Proof of Proposition 7.1.2) To see that Q_H is not convex, note that the form of Q_H implies that $\text{Im } Q_H|_{\mathbb{S}^3}$ is a flat base for $\text{Im } Q_H$. Denote this base by B .

Now, $Q_H|_{\mathbb{S}^3}$ is just the composition of the Hopf map $\overline{Q}_H|_{\mathbb{S}^3}$ with the inclusion $i_{H_{e_4}^1} : \mathbb{R}^3 \hookrightarrow H_{e_4}^1$, which sends (x_1, x_2, x_3) to $(x_1, x_2, x_3, 1)$. It follows that $B = \{(x_1, x_2, x_3, 1) \mid (x_1, x_2, x_3) \in \mathbb{S}^2\}$. B , therefore, is not convex. It follows from Lemma 3.4.17 that $\text{Im } Q_H$ is not convex either.

To see that Q_H is boundary-convex, note that $\text{Conv}(B) = \{(x_1, x_2, x_3, 1) \mid (x_1, x_2, x_3) \in \mathbb{D}^3\}$, and therefore that $\partial_{H_{e_4}^1} \text{Conv}(B) \subset B$ (in fact, $\partial_{H_{e_4}^1} \text{Conv}(B) = B$). By Proposition 3.4.19, $\partial_{\mathbb{R}^4} \text{Conv}(\text{Im } Q_H) = \text{Cone}(\partial_{H_{e_4}^1} \text{Conv}(B)) \subset \text{Cone}(B) = \text{Im } Q_H$. Thus, Q_H is boundary-convex. \square

Of course, the Hopf map is somewhat special. It is not currently known whether the phenomenon exhibited by Q_H occurs in dimensions greater than 4, or whether it is exhibited by other quadratic maps in \mathcal{Q}_4 that are inequivalent to Q_H .

7.2 INDEFINITE AND SEMIDEFINITE QUADRATIC MAPS

Chapter 4 focused all of its attention on definite quadratic maps. This section briefly discusses indefinite and semidefinite maps.

Definition 7.2.1. Let V and W denote finite-dimensional real inner product spaces. Let Q denote a quadratic map in $\mathcal{Q}(V, W)$ and let $\psi \in \mathcal{L}(W, \text{Sym}(V))$ denote its ALM. Q is **semidefinite** if $\psi(W) \subset \text{Sym}(V)$ is a semidefinite subspace of $\text{Sym}(V)$. Q is **indefinite** if $\psi(W)$ is an indefinite subspace of $\text{Sym}(V)$.

INDEFINITE MAPS

Indefinite quadratic maps can be characterized by the convexifications of their images.

Proposition 7.2.2. Let $Q \in \mathcal{Q}(V, W)$ be a quadratic map with ALM ψ . Then, Q is indefinite if and only if $\text{Conv}(\text{Im } Q) = (\ker \psi)^\perp$.

Proof. Proposition 4.2.13, which says that $(\text{Im } Q)^* = K_\psi$, applies to all quadratic maps, not just definite ones. Thus, if Q is indefinite, then $(\text{Im } Q)^* = (\text{Conv}(\text{Im } Q))^* = \ker \psi$. So, by Proposition 3.4.34, $\text{Conv}(\text{Im } Q) = (\ker \psi)^\perp$. \square

Corollary 7.2.3. *Let $Q \in \mathcal{Q}(V, W)$ be an indefinite map with ALM ψ . Then, if $\ker \psi = \mathbf{o}$, $\text{Conv}(\text{Im } Q) = W$.*

It follows from Corollary 7.2.3 that if Q is indefinite, convex, and $\ker \psi = \{\mathbf{o}\}$, then $\psi(w^\perp)$ must be a nice subspace of $\text{Sym}(V)$ for every nonzero $w \in W$. However, this is not a sufficient condition for convexity, even for quadratic maps mapping into \mathbb{R}^3 .

Example 7.2.4. *Let $Q \in \mathcal{Q}_3$ be the following indefinite, nondegenerate quadratic map (with ALM ψ),*

$$Q : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} (x_1)^2 - (x_2)^2 \\ 2x_1x_2 \\ (x_1)^2 + (x_2)^2 - (x_3)^2 \end{pmatrix}$$

Calabi's Niceness Theorem (Theorem 3.5.9) implies that $\psi(u^\perp)$ is a nice subspace of Sym_3 , for every nonzero $u \in \mathbb{R}^3$. However, the image of Q is not convex, for Q is not surjective. The image of Q does not contain the hyperplane e_1^\perp . In fact, a straightforward calculation can show that

$$\text{Im } Q \cap e_1^\perp = \left\{ \begin{pmatrix} \mathbf{o} \\ a \\ b \end{pmatrix} \mid a, b \in \mathbb{R}, b \leq a \right\}$$

It is not currently known whether there exist convex indefinite maps that map into \mathbb{R}^n for $n \geq 3$.

SEMIDEFINITE MAPS

Nondegenerate semidefinite quadratic maps can also be characterized by the convexifications of their images. The following proposition distinguishes semidefinite maps from definite ones according to this: the convexification of the image of a definite map cannot contain a subspace, whereas the convexification of the image of semidefinite map must.

Proposition 7.2.5. *Let $Q \in \mathcal{Q}(V, W)$ be a nondegenerate map satisfying $Q^{-1}(\mathbf{o}) = \{\mathbf{o}\}$, with $ALM \psi$. Then, Q is semidefinite if and only if, for some nonzero $u \in W$, $Im Q$ is contained in the closed half-space H_u^+ and if $P \subset Conv(Im Q)$ for some nonzero linear subspace $P \subset u^\perp$.*

The proof of Proposition 7.2.5 calls on a pair of lemmas.

Lemma 7.2.6. *Let $n \in \mathbb{Z}^+$. Let $M \in S_n^+$ have rank $k \leq n$ and suppose that $Im M = \mathbb{R}^k$ (so, M has the following block form,*

$$M = \begin{pmatrix} A & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{pmatrix}$$

where $A \in \mathring{S}_k^+$). Let $M' \in Sym_n$ be a matrix whose lower-right $(n - k) \times (n - k)$ block is positive semidefinite and nonzero (so, M' has the following block form,

$$M' = \begin{pmatrix} A' & B \\ (B)^T & C \end{pmatrix}$$

where $C \in S_{n-k}^+$ and $C \neq \mathbf{o}$. Then, for all sufficiently large $t \in \mathbb{R}$, $tM + M'$ is positive semidefinite and $rank tM + M' > rank M$.

Proof. Let $Q(t) \in Sym_k$ denote the matrix $tA + A'$ (the upper-left $k \times k$ block of $tM + M'$). Note that the minimum eigenvalue of $Q(t)$ can be made arbitrarily large by letting t be sufficiently large itself. Thus, $Q(t)^{-1}$ can be brought arbitrarily close to $\mathbf{o} \in Sym_k$, again by choosing t appropriately.

If elements of \mathbb{R}^n are written as

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$, then,

$$\begin{pmatrix} x^T & y^T \end{pmatrix} (tM + M') \begin{pmatrix} x \\ y \end{pmatrix} \quad (7.2.1)$$

$$= x^T Qx + 2x^T By + y^T Cy$$

$$= (x + Q(t)^{-1}By)^T Q(t) (x + Q(t)^{-1}By) + y^T (C - B^T Q(t)^{-1}B) y \quad (7.2.2)$$

If t is chosen to be large enough for $Q(t)$ to be positive definite, then the first term in (7.2.2) is nonnegative. Moreover, if t is sufficiently large, then $Q(t)^{-1}$ will be close enough to 0 $\in \text{Sym}_k$ for $C - BQ(t)^{-1}B$ to be positive semidefinite in Sym_{n-k} . Thus, for all sufficiently large t , the sum in (7.2.2) is nonnegative for all $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$. This proves the first assertion in the lemma.

To see that $\text{rank } tM + M' > \text{rank } M$ for all sufficiently large t , note that the image of $tM + M'$ is equal to the sum,

$$\begin{aligned} & \text{Im} \begin{pmatrix} Q(t) \\ B \end{pmatrix} + \text{Im} \begin{pmatrix} B \\ C \end{pmatrix} \\ &= \begin{pmatrix} Q(t) & 0 \\ 0 & Q(t) \end{pmatrix} \cdot \text{Im} \begin{pmatrix} I_k \\ Q(t)^{-1}B \end{pmatrix} + \text{Im} \begin{pmatrix} B \\ C \end{pmatrix} \end{aligned} \quad (7.2.3)$$

where the first term in (7.2.3) denotes the image of the matrix

$$\begin{pmatrix} I_k \\ Q(t)^{-1}B \end{pmatrix}$$

under the matrix

$$\begin{pmatrix} Q(t) & 0 \\ 0 & Q(t) \end{pmatrix}$$

By making $Q(t)$ sufficiently large, and $Q^{-1}B$ thereby sufficiently close to 0, the first term in (7.2.3) can be brought arbitrarily close to \mathbb{R}^k . The sum in (7.2.3) will

then have dimension greater than k . □

The following corollary is a coordinate-free version of the foregoing lemma. It invokes the restriction operator R_L defined in Remark 5.2.4.

Corollary 7.2.7. *Let V and W be finite-dimensional real inner product spaces. Let $M \in S^+(V)$ and let $L = \ker M \subset V$. Let $M' \in \text{Sym}(V)$ be such that $R_L(M')$ is nonzero and lies in $S^+(L)$. Then, for all sufficiently large $t \in \mathbb{R}$, $tM + M'$ is in $S^+(V)$ and $\text{rank } tM + M' > \text{rank } M$.*

The following lemma is the last of the two that are needed for the proof of Proposition 7.2.5.

Lemma 7.2.8. *Let V be a finite-dimensional real inner product space. Let $P \subset \text{Sym}(V)$ be a semidefinite subspace and let $M \in P \cap S^+(V)$ have maximal rank among those elements of $P \cap S^+(V)$. Let $L = \ker M$. If the subspace $R_L(P)$ of $\text{Sym}(L)$ is nonzero, then it is an indefinite subspace.*

Proof. Suppose that $R_L(P)$ is a semidefinite subspace of $\text{Sym}(L)$ (the proof is the same if $R_L(P)$ is a definite subspace). Let $M' \in \text{Sym}(V)$ be such that $R_L(M')$ is nonzero, and lies in $S^+(L)$. Then, by Corollary 7.2.7, for all sufficiently large $t \in \mathbb{R}$, $tM + M'$ is positive semidefinite and $\text{rank } tM' + M > \text{rank } M$. This contradicts the choice of M (whose rank was maximal). Thus, $R_L(P)$ is indefinite. □

Proof. (Proof of Proposition 7.2.5) Let $P = \text{Im } \psi$ and let $\psi(u) \in P \cap S^+(V)$ have maximal rank among those elements of $P \cap S^+(V)$. Let $L = \ker \psi(u)$. By the fundamental relation (4.1.2), the image of the restriction $Q|_L : L \rightarrow W$ is contained in u^\perp . Thus, the image of $Q|_L$ is the same as that of $Q' = \pi_{u^\perp} \circ Q|_L : L \rightarrow u^\perp$, where $\pi_{u^\perp} : W \rightarrow u^\perp$ is the orthogonal projection onto u^\perp . Note that the assumption $Q^{-1}(o) \neq \{o\}$ implies that $\text{Im } Q' \neq \{o\}$.

By Proposition 5.2.5, the ALM of Q' is $\psi' = R_L \circ \psi \circ i_{u^\perp}$, where $i_{u^\perp} : u^\perp \hookrightarrow W$ is the inclusion map. Thus, $\text{Im } \psi' \subset \text{Sym}(L)$ is contained in $R_L(\text{Im } \psi) =$

$R_L(P)$, which, by Lemma 7.2.8, is an indefinite subspace of $\text{Sym}(L)$. Moreover, by the last sentence of the previous paragraph, $\text{Im } \psi' \neq \{0\}$.

It now follows from Proposition 7.2.2 that $\text{Conv}(\text{Im } Q')$ (and therefore $\text{Conv}(\text{Im } Q)$) contains a nonzero subspace of u^\perp . \square

7.3 AN ALTERNATIVE DESCRIPTION OF $\text{Conv}(\text{Im } Q)$

Proposition 4.4.6 states that the convexification of the image of a definite quadratic map Q is the dual cone K_ψ^* . This section provides a more elementary description of $\text{Conv}(\text{Im } Q)$ that applies to any quadratic map.

Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a quadratic map defined by the matrices $M_1, \dots, M_m \in \text{Sym}_n$. Recall that, for $x \in \mathbb{R}^n$,

$$\langle x, M_i x \rangle_{\mathbb{R}^n} = \langle M_i, x \otimes x \rangle_{\text{Sym}_n} \quad (7.3.1)$$

Let $L : \text{Sym}_n \rightarrow \mathbb{R}^m$ be the linear map defined by

$$L(M) = \begin{pmatrix} \langle M, M_1 \rangle_{\text{Sym}_n} \\ \vdots \\ \langle M, M_m \rangle_{\text{Sym}_n} \end{pmatrix}$$

The identity (7.3.1) implies that the image of Q coincides with the image of the restriction of L to $S_{n,1}^+$, the set of rank 1 positive semidefinite matrices in Sym_n : $\text{Im } Q = \text{Im } L|_{S_{n,1}^+}$. Thus,

$$\begin{aligned} \text{Conv}(\text{Im } Q) &= \text{Conv}(L(S_{n,1}^+)) \\ &= L(\text{Conv}(S_{n,1}^+)) \\ &= L(S_n^+) \\ &= \left\{ \begin{pmatrix} \langle M, M_1 \rangle_{\text{Sym}_n} \\ \vdots \\ \langle M, M_m \rangle_{\text{Sym}_n} \end{pmatrix} : M \in S_n^+ \right\} \end{aligned} \quad (7.3.2)$$

The second equality follows from equation (3.3.1), and the third from the fact that every positive semidefinite matrix is a convex combination of rank 1 positive semidefinite matrices. Equation (7.3.2) is the description of $\text{Conv}(\text{Im } Q)$ alluded to at the outset of this section.

There are now two linear maps associated with each quadratic map $Q \in \mathcal{Q}(\mathbb{R}^n, \mathbb{R}^m)$: $L : \text{Sym}_n \rightarrow \mathbb{R}^m$ as defined above, and $\psi : \mathbb{R}^m \rightarrow \text{Sym}_n$ the ALM of Q . The two are closely related; they are adjoints of one another. This follows from a simple calculation:

$$\begin{aligned} \langle L(M), e_i \rangle_{\mathbb{R}^m} &:= \langle M, M_i \rangle_{\text{Sym}_n} \\ &= \langle M, \psi(e_i) \rangle_{\text{Sym}_n} \\ &= \langle \psi^*(M), e_i \rangle_{\mathbb{R}^m} \end{aligned}$$

Thus, $\text{Conv}(\text{Im } Q) = L(S_n^+) = \psi^*(S_n^+)$.

If Q is a definite map, the description $\text{Conv}(\text{Im } Q) = \psi^*(S_n^+)$ can be reconciled with Proposition 4.4.6 with the help of Corollary 3.4.45. Corollary 3.4.45 says that $K_\psi = (\psi^*(S_n^+))^*$, where ψ is the ALM of Q . If Q is a definite quadratic map, then K_ψ is a definite cone in \mathbb{R}^m . Thus, by Theorem 3.4.36, $(\psi^*(S_n^+))^{**} = \text{Conv}(\psi^*(S_n^+)) = \psi^*(S_n^+)$, the final equality holding because $\psi^*(S_n^+)$ is convex. It finally follows that K_ψ , $\psi^*(S_n^+)$ and $\text{Conv}(\text{Im } Q)$ are all equal when Q is definite.

7.4 A NOTE ON NONCONVEX QUADRATIC MAPS

This section proves the following proposition.

Proposition 7.4.1. *Let $n, m \in \mathbb{Z}^+$, with $m \leq n$. The set of nondegenerate, definite, nonconvex quadratic maps from \mathbb{R}^n to \mathbb{R}^m is open in $\text{Def}_{n,m}$ (the subset of $\mathcal{Q}_{n,m}$ consisting of definite quadratic maps).*

Proposition adds a modicum of detail to the picture of $\text{Def}_{n,m}$ that was painted in Chapters 4 and 5. Within $\text{Def}_{n,m}$ is the open, dense subset of nondegenerate

definite maps. In turn, the set of nondegenerate, definite maps decomposes into the disjoint union of the open set of nonconvex nondegenerate definite maps and the relatively closed set of convex nondegenerate definite maps. Of course, the set of convex nondegenerate definite maps contains the open set of round maps.

Proposition 7.4.1 requires two lemmas. The first lemma resurrects the notation $\mathcal{Z}(M)$ from Definition 3.5.6, where $M \in \text{Sym}_n$. The forthcoming discussion will profit from the additional notation $\tilde{\mathcal{Z}}_M := \mathcal{Z}_M \cap S^{n-1}$. Recall in addition the notation \mathcal{Z}_P of Definition 3.5.6, where $P \subset \text{Sym}_n$ is a linear subspace.

Lemma 7.4.2. *Let $n \in \mathbb{Z}^+$. Suppose $M \in \text{Sym}_n$ is not definite, so that $\tilde{\mathcal{Z}}_M \neq \emptyset$. Then, given an open neighbourhood $U \subset S^{n-1}$ containing $\tilde{\mathcal{Z}}_M$, there exists an open neighbourhood $V \subset \text{Sym}$ containing M with the following property: $M' \in V \Rightarrow \tilde{\mathcal{Z}}_{M'} \subset U$.*

Proof. For $M \in \text{Sym}_n$, let $q_M : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the map $x \mapsto \langle x, Mx \rangle$. Since $S^{n-1} - U$ is compact, the image $q_M(S^{n-1} - U)$ is closed in \mathbb{R} . Furthermore, since $\tilde{\mathcal{Z}}_M = q_M^{-1}(\{0\}) \subset U$, the image $q_M(S^{n-1} - U)$ does not contain $0 \in \mathbb{R}$. Thus, there exists an interval $(-\delta, \delta)$ with $\delta > 0$ which is not contained in $q_M(S^{n-1} - U)$. It follows that $q_M^{-1}((-\delta, \delta)) \cap S^{n-1} \subset U$.

Let $\|M\|$ denote the sup-norm on Sym : $\|M\| = \sup_{\|x\|=1} \|Mx\|_{\mathbb{R}^n}$. Let $V = B_{\frac{\delta}{2}}(M) \subset \text{Sym}$, the open ball of radius $\frac{\delta}{2}$ centered at M (under the sup-norm). For $M' \in V$ and $x' \in \tilde{\mathcal{Z}}_{M'}$,

$$\begin{aligned}
q_M(x') &= \langle x', Mx' \rangle \\
&= \langle x', (M - M')x' \rangle && \text{since } x' \in \tilde{\mathcal{Z}}_{M'} \\
&\leq \|x'\|_{\mathbb{R}^n} \|(M - M')x'\|_{\mathbb{R}^n} && \text{(Cauchy-Schwarz)} \\
&= \|(M - M')x'\|_{\mathbb{R}^n} && \text{since } x' \in S^{n-1} \\
&\leq \|M - M'\| && \text{by definition of the sup-norm} \\
&< \frac{\delta}{2}
\end{aligned}$$

So, $x' \in q_M^{-1}(\frac{\delta}{2}) \subset U$ for all $x' \in \tilde{\mathcal{Z}}_{M'}$, for all $M' \in V$. That is,

$$M' \in V \Rightarrow \tilde{Z}_{M'} \subset U.$$

□

The second lemma needed for Proposition 7.4.1 requires some basic remarks on Grassmann manifolds (for a reference, see [28]). Let V and W be finite-dimensional, real vector spaces. The *Grassmann manifold*, $\text{Gr}_k(V)$ is the set of k -dimensional subspaces of V . Every injective map $\psi : V \rightarrow W$ induces a map $\bar{\psi} : \text{Gr}_k(V) \rightarrow \text{Gr}_k(W)$ that takes a k -dimensional subspace $P \subset V$ to its image $\psi(P) \subset W$ under ψ . Moreover, the map $\psi \mapsto \bar{\psi}$ is an embedding of the set of injective linear maps from V to W into the set of continuous maps from $\text{Gr}_k(V)$ to $\text{Gr}_k(W)$ (which is endowed with the compact-open topology). Finally, note that there is a projection π onto $\text{Gr}_k(V)$ from the subset of $V \times \cdots \times V$ (the k -fold product) consisting of linearly independent k -tuples; the projection sends each such tuple to the subspace that the tuple spans. The projection π induces a quotient topology on $\text{Gr}_k(V)$, and in that topology π is an open map.

The following lemma says that an indefinite subspace of Sym_n that is not nice can be perturbed slightly (in a sense specified by the lemma) without losing the property of being indefinite and not nice.

Lemma 7.4.3. *Let $k, n \in \mathbb{Z}^+$ with $1 \leq k \leq n$. The set N_k of not-nice k -dimensional subspaces of Sym_n is an open set in $\text{Gr}_k(\text{Sym}_n)$.*

Proof. If $k = 1$, then $\text{Gr}_1(\text{Sym}) = \mathbb{P}(\text{Sym})$, the projectivization of Sym_n . $P \in \mathbb{P}(\text{Sym}_n)$ is not nice if and only if $P = \text{span}\{M\}$ where $M \in \text{Sym}_n$ is positive definite. Every positive definite $M \in \text{Sym}_n$ has an open neighbourhood $U \subset \text{Sym}$ consisting entirely of positive definite elements. Moreover, U descends to an open neighbourhood $\pi(U)$ of P under the quotient map $\pi : \text{Sym} \rightarrow \mathbb{P}(\text{Sym})$. The neighbourhood $\pi(U)$ is then an open neighbourhood of P disjoint from the set of nice 1-dimensional subspaces of Sym_n . This proves the lemma for $k = 1$.

Suppose now that $k > 1$. Let $\{M_i : i = 1, \dots, k\}$ be a basis of P consisting entirely of indefinite elements (such bases exist for any subspace of Sym_n whose dimension exceeds 1). Because P is not nice, the sets \tilde{Z}_{M_i} , $1 \leq i \leq k$ have empty

intersection. In particular,

$$\mathcal{Z}_1 \cap \bigcup_{i=2}^k \mathcal{Z}_i = \emptyset$$

(where now, for notational convenience, \mathcal{Z}_i denotes \mathcal{Z}_{M_i}). Both \mathcal{Z}_1 and $\bigcup_{i=2}^k \mathcal{Z}_i$ are closed and therefore compact subspaces of S^{n-1} . Because they are disjoint, and because S^{n-1} is a normal topological space, there exist disjoint open neighbourhoods $U, V \subset S^{n-1}$, with $\tilde{\mathcal{Z}}_1 \subset U$ and $\bigcup_{i=2}^k \tilde{\mathcal{Z}}_i \subset V$.

By Lemma 7.4.2, there exists, for each $i = 1, \dots, k$ an open neighbourhood $W_i \subset \text{Sym}_n$ containing M_i , and which is such that $M' \in W_1 \Rightarrow \mathcal{Z}_{M'} \subset U$ and $M' \in W_i \Rightarrow \mathcal{Z}_{M'} \subset V$ for $2 \leq i \leq k$. The W_i can be chosen such that all k -tuples from $W_1 \times W_2 \times \dots \times W_k$ are linearly independent. If chosen in this way, Then, $\pi(W_1 \times \dots \times W_k)$ is an open set in $\text{Gr}_k(\text{Sym})$ containing P and containing *no* nice subspaces. This proves the lemma.

For an alternative proof, consider a convergent sequence $P_i \rightarrow P$ in $\text{Gr}_k(\text{Sym})$ in which each P_i is a nice subspace. The niceness of P_i implies the existence of an $x_i \in S^{n-1}$ such that $x_i \in \tilde{\mathcal{Z}}_{P_i}$. Since $x_i \in S^{n-1}$, there is a convergent subsequence $x_{i_q} \rightarrow x \in S^{n-1}$.

The following argument will show that $x \in \mathcal{Z}_P$, in which case P is nice; this also proves the lemma. To see that $x \in \mathcal{Z}_P$, choose a basis

$B_i = \{M_{i,j} : j = 1, \dots, k\}$ of P_i for each $i \in \mathbb{Z}^+$. Then, $P_i \rightarrow P \Leftrightarrow M_{i,j} \rightarrow N_j$ where $\{N_j : j = 1, \dots, k\}$ is a basis for P . Then, $\langle x_{k_q}, M_i^{k_q} x_{k_q} \rangle \rightarrow \langle x, N_i x \rangle$, which (since $\langle x_{k_q}, M_i^{k_q} x_{k_q} \rangle = 0$) implies that $\langle x, N_i x \rangle = 0$ for $1 \leq i \leq k$ and therefore that $\langle x, Nx \rangle = 0$ for all $N \in P$. \square

To prove Proposition 7.4.1, let $\psi : \mathbb{R}^m \rightarrow \text{Sym}_n$ be the ALM of a nonconvex, nondegenerate, definite quadratic map $Q \in \mathcal{Q}_{n,m}$. Because Q is nondegenerate, $\ker \psi = \{0\}$. It follows that ψ induces a map $\bar{\psi} : \text{Gr}_{m-1}(\mathbb{R}^m) \rightarrow \text{Gr}_{m-1}(\text{Sym}_n)$. Because Q is definite and nonconvex, $\text{Im } \psi$ contains an indefinite, $n-1$ -dimensional subspace of Sym_n that is not nice (this is by Theorem 4.4.12). Thus, the image of $\bar{\psi}$ intersects the subset $N_{m-1} \subset \text{Gr}_{m-1}(\text{Sym}_n)$ consisting of

$m - 1$ -dimensional subspaces that are indefinite and not nice. By Lemma 7.4.3, N_{m-1} is open. Therefore, there is an open set in the set of continuous maps from $\text{Gr}_{m-1}(\mathbb{R}^m)$ to $\text{Gr}_{m-1}(\text{Sym}_n)$ containing $\bar{\psi}$, each element $\bar{\psi}'$ of which also satisfies $\text{Im } \bar{\psi}' \cap N_{m-1} \neq \emptyset$. It follows that there is an open neighbourhood of ψ in $\mathcal{L}(\mathbb{R}^m, \text{Sym}_n)$ each of whose members is the ALM of a nonconvex, definite, nondegenerate quadratic map.

7.5 A WORD ON THE PAPER BY GUTKIN ET AL.

This section discusses the paper [17], which was mentioned in passing in Section 2.1. The main result in [17] bears a resemblance to the Roundness Theorem; this section explores the relationship between the two results. I should note that I discovered the paper [17] only after I had completed the work in this thesis.

This section uses an amalgam of notation. Some is taken from [17] and the rest is taken from this thesis. For example, as is done in [17], $\lambda_1(M)$ will denote the smallest eigenvalue of $M \in \text{Sym}_n$, and $E_1(M) \subset \mathbb{R}^n$ will denote the corresponding eigenspace of M (actually, $\lambda_1(M)$ is used in [17] to denote the largest eigenvalue of M ; this discrepancy is explained below in Remark 7.5.6).

In keeping with the notation of Section 2.1, a quadratic map $W : \mathbb{R}^{2n} \rightarrow \mathbb{R}^m$ will be called *complex* if its defining matrices lie in the subspace $\mathcal{C} \subset \text{Sym}_{2n}$ (which was defined in Section 2.1),

$$\mathcal{C} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A \in \text{Sym}_n, B \in \mathfrak{so}(n) \right\} \subset \text{Sym}_{2n}$$

The main theorem in [17] is reproduced below as Theorem 7.5.2. Theorem 7.5.2 is stated in terms that differ from those used in [17]; this is done to make the theorem more readily comparable to the Roundness Theorem.

Remark 7.5.1. In this chapter, \mathbb{R}^{m-1} will always be identified with $e_m^\perp \subset \mathbb{R}^m$. ■

Theorem 7.5.2. *Let $n \in \mathbb{Z}^+$. Let H_1, \dots, H_{m-1} be elements of $\mathcal{C} \subset \text{Sym}_{2n}$ and let*

$W : \mathbb{C}^n \rightarrow \mathbb{R}^m$ be the complex quadratic map defined by

$$x \in \mathbb{R}^{2n} \mapsto \begin{pmatrix} \langle x, H_1 x \rangle_{\mathbb{R}^{2n}} \\ \vdots \\ \langle x, H_{m-1} x \rangle_{\mathbb{R}^{2n}} \\ \|x\|^2 \end{pmatrix}$$

Let ψ_W denote the ALM of W , and let $W_S : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}^{m-1}$ denote the map $\pi_m \circ (W|_{\mathbb{S}^{2n-1}})$, where $\pi_m : \mathbb{R}^m \rightarrow e_m^\perp = \mathbb{R}^{m-1}$ is the orthogonal projection of \mathbb{R}^m onto e_m^\perp .

Suppose that the following conditions hold,

- If $\bar{u} \in \mathbb{R}^{m-1}$, then the multiplicity of $\lambda_1(\psi_W(\bar{u}))$, the largest eigenvalue of $\psi_W(\bar{u})$, is the same for all nonzero $\bar{u} \in \mathbb{R}^{m-1}$.
- The union

$$\bigcup_{\bar{u} \in \mathbb{R}^{m-1}, \bar{u} \neq 0} E_1(\psi(\bar{u}))$$

is not equal to all of \mathbb{R}^{2n} .

Then, $\text{Im } W_S \subset \mathbb{R}^{m-1}$ is convex.

Remark 7.5.3. The authors of [17] note that the second hypothesis in Theorem 7.5.2 holds only if $m = 2n - 1$, a case that is not covered by the Roundness Theorem. The focus of this section will be on the first hypothesis in Theorem 7.5.2. ■

Theorem 7.5.2 can be translated in a way that makes it more proximate to the Roundness Theorem. To do so, note that if $u = (\bar{u}, \lambda) \in \mathbb{R}^m$, with $\bar{u} \in \mathbb{R}^{m-1}$ and $\lambda \in \mathbb{R}$, then,

$$\psi_W(u) = \psi_W(\bar{u}) + \lambda I_{2n}$$

where I_{2n} denotes the identity map on \mathbb{R}^{2n} . Now, for $\bar{u} \in \mathbb{R}^{m-1}$, the matrix

$$\psi_W(\bar{u}) - \lambda_1(\psi_W(\bar{u}))I_{2n}$$

is positive semidefinite and has the nonzero kernel, $E_1(\psi_W(\bar{u}))$ (this is true for any symmetric matrix, not just $\psi_W(\bar{u})$). It follows that, for all nonzero $\bar{u} \in \mathbb{R}^{m-1}$, the element

$$u = (\bar{u}, \lambda_1(\psi_W(\bar{u}))) \in \mathbb{R}^m$$

lies in the boundary of the pullback cone $K_{\psi_W}: u \in \partial_{\mathbb{R}^m} K_{\psi_W}$. The converse is true as well: if $u = (\bar{u}, -\lambda)$ lies in $\partial_{\mathbb{R}^m} K_{\psi_W}$, then $\psi_W(\bar{u}) - \lambda I_{2n}$ is positive semidefinite and has a nonzero kernel. It follows that $\lambda = \lambda_1(\psi_W(\bar{u}))$ and that $E_1(\psi_W(\bar{u})) = \ker(\psi_W(\bar{u}) - \lambda I_{2n})$.

Given the foregoing observations, Theorem 7.5.2 can be rephrased as follows,

Theorem 7.5.4. *Let $n, H_1, \dots, H_{m-1}, W, \psi_W$ and W_S be defined as they were in Theorem 7.5.2. Suppose that the following conditions hold,*

- $\dim \ker \psi_W(u)$ is the same for all nonzero $u \in \partial_{\mathbb{R}^m} K_{\psi_W}$
- The union

$$\bigcup_{u \in \partial_{\mathbb{R}^m} K_{\psi_W}, u \neq 0} \ker \psi(u)$$

is not equal to all of \mathbb{R}^{2n} .

Then, W_S is convex.

In this alternative form, Theorem 7.5.2 bears a stronger likeness to the Roundness Theorem. After all, the Roundness Theorem also says that, given a quadratic map $Q: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (with ALM ψ_Q), the image of the map $Q_S := \pi_m \circ (Q|_{\mathbb{S}^{n-1}})$ is convex so long as $\dim \ker \psi_Q(u)$ is constant (and equal to 1) for all nonzero $u \in \partial_{\mathbb{R}^m} K_{\psi_Q}$.

However, Theorem 7.5.2 and the Roundness Theorem are inequivalent. In fact, neither theorem implies the other. For, on one hand, Theorem 7.5.2 applies only to those quadratic maps in $\mathcal{Q}_{2n,m}$ whose defining matrices lie in the subspace $\mathcal{C} \subset \text{Sym}_{2n}$, whereas the Roundness Theorem applies to all quadratic maps in $\mathcal{Q}_{n,m}$ (except possibly for $n = m + 1$). On the other hand, while the Roundness Theorem says that the equality $\dim \ker \psi_Q(u) = 1$ for all nonzero $u \in K_{\psi_Q}$ is sufficient for the convexity of Q_S , Theorem 7.5.2 says that the constancy of $\dim \ker \psi_W(u)$ alone is sufficient.

This section concludes with a brief comparison of the proof of Theorem 7.5.2 and the proof of the Roundness Theorem. Section 5.4 showed that if $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a round quadratic map, then the restriction of Q_S to

$$\mathbb{S}^{n-1} \cap \bigcup_{u \in \partial_{\mathbb{R}^m} K_{\psi_Q}, u \neq 0} \ker \psi_Q(u) \quad (7.5.1)$$

is a double cover of $\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)$. A double cover, for reasons soon to be apparent, can be thought of as an \mathbb{S}^0 -bundle (a fiber-bundle whose fibers are \mathbb{S}^0).

A similar observation undergirds the proof of Theorem 7.5.2, namely that the restriction of W_S to the subset

$$\mathcal{S} = \mathbb{S}^{2n-1} \cap \bigcup_{u \in \partial_{\mathbb{R}^m} K_{\psi_W}, u \neq 0} \ker \psi_W(u) = \mathbb{S}^{2n-1} \cap \bigcup_{u \in \partial_{\mathbb{R}^m} K_{\psi_W}, u \neq 0} E_1(\psi_W(u)) \quad (7.5.2)$$

is a $(k - 1)$ -sphere bundle of $\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } W_S)$, where k is the common multiplicity of $\lambda_1(\psi_W(\bar{u}))$ for $\bar{u} \in \mathbb{R}^{m-1}$, $\bar{u} \neq 0$ (a $(k - 1)$ -sphere bundle being a fiber-bundle whose fibers are \mathbb{S}^{k-1}).

The proof of the Roundness Theorem and the proof of Theorem 7.5.2 differ slightly in their utilization of the foregoing observations about the sets (7.5.1) and (7.5.2). The proof of the Roundness Theorem notes that the set (7.5.1) is actually a disjoint union of two smoothly embedded copies of \mathbb{S}^{m-2} in \mathbb{S}^{n-1} , each of which bounds one of a pair of disjoint, smoothly embedded disks \mathbb{D}^{m-1} .

$\subset \mathbb{S}^{n-1}$. From there, the proof shows that the restriction of Q_S to D_1 is a surjection onto $\text{Conv}(\text{Im } Q_S)$ (the failure of Q_S to be surjective would lead to the construction of a restriction from $\text{Conv}(\text{Im } Q_S)$ to $\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } Q_S)$, a known impossibility).

The proof of Theorem 7.5.2 exploits the nature of (7.5.2) in a slightly different way. The authors of [17] show that if W_S is not a surjection onto $\text{Conv}(\text{Im } W_S)$, then the restriction $W_S|_{\mathcal{S}}$ is homotopic to a constant map. This, as the authors point out, is also an impossibility; the projection map of a sphere-bundle over a sphere (in this case, over $\partial_{\mathbb{R}^{m-1}} \text{Conv}(\text{Im } W_S)$, which is homeomorphic to a sphere), cannot be homotopic to a constant map.

The means by which the authors of [17] arrive at their conclusions about the set (7.5.2) differ from those by which this thesis studies the set (7.5.1). Section 5.4 studied the set (7.5.1) via the boundary of the pull-back cone K_{ψ_Q} . On the other hand, the authors of [17] studied the set (7.5.2) using *support functions*.

Definition 7.5.5. Let $A \subset \mathbb{R}^m$ be a compact, convex set. The **support function** of A is the function $s_A : \mathbb{R}^m \rightarrow \mathbb{R}$ defined by,

$$s_A(x) = \max_{a \in A} \langle a, x \rangle_{\mathbb{R}^n}$$

The analysis of (7.5.2) using support functions will not be discussed here, other than to say that it is, in my opinion, less accessible than the dual cone approach to studying (7.5.1).

Remark 7.5.6. The largest eigenvalue of $\psi_W(\bar{u})$ is equal to the smallest eigenvalue of $\psi_W(-\bar{u})$. The first condition in Theorem 7.5.2 is therefore equivalent to the condition that the multiplicity of the *smallest* eigenvalue of $\psi_W(\bar{u})$ be constant over $\mathbb{R}^{m-1} - \{0\}$; the latter is the condition used in [17]. The change was made to make the result of [17] more easily comparable to the Roundness Theorem. ■

Part II

The Controllability of Control-Coupled Systems

8

The Controllability of Control-Coupled Systems

8.1 INTRODUCTION

This chapter establishes controllability criteria for invariant, bilinear control systems that are defined on a special type of Lie group: those that are the direct product of two or more Lie groups, at least one of which is simple. For reasons that were explained in Chapter 1, control systems of this sort will be referred to here as *control-coupled* systems.

Well-known controllability criteria for invariant bilinear systems on general Lie groups already exist [22]. Indeed, they are the instruments by which the criteria in this chapter are derived. The point of this chapter, however, is to demonstrate that when applied to control-coupled systems, the general criteria of [22]

become considerably more precise (see, for example, Corollaries 8.4.1 and 8.4.2).

Among the systems of interest in this chapter are those that are defined on Lie groups with the form $\mathcal{G} \times \mathcal{G}$ where \mathcal{G} is a simple Lie group. Systems of this type will be written as

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) \begin{pmatrix} X_i \\ Y_i \end{pmatrix} (\gamma(t)) \quad (8.1.1)$$

where $\gamma : [0, \infty) \rightarrow \mathcal{G} \times \mathcal{G}$ is a differentiable curve, the u_i are controls (say, piecewise continuous functions), X_i and Y_i elements of the Lie algebra \mathfrak{g} of \mathcal{G} (thought of as here as the Lie algebra of left-invariant vector fields on \mathcal{G}) and where

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$$

denotes an element of $\mathfrak{g} \oplus \mathfrak{g}$, the Lie algebra of $\mathcal{G} \times \mathcal{G}$ (this notation mimics the practice of writing elements of $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ as column vectors). The notation

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} (\gamma(t))$$

is meant to denote the vector field (X_i, Y_i) on $\mathcal{G} \times \mathcal{G}$, evaluated at $\gamma(t)$.

System (8.1.1) is controllable only if the same is true of each of its component systems,

$$\begin{aligned} \gamma_1(t) &= \sum_{i=1}^m u_i(t) X_i(\gamma_1(t)) \quad \text{and} \\ \gamma_2(t) &= \sum_{i=1}^m u_i(t) Y_i(\gamma_2(t)) \end{aligned} \quad (8.1.2)$$

In the event that the systems (8.1.2) are controllable, system (8.1.1) will be called *component-wise controllable*. Component-wise controllability does not always

guarantee the controllability of the full coupled system (8.1.1). For example, if $X_i = Y_i$ for all $i = 1, \dots, m$, then, even though each of the component systems (8.1.2) might be controllable, the coupled system (8.1.1) will always be bound to cosets of the diagonal subgroup $\Delta = \{(g, g) : g \in \mathcal{G}\}$ of $\mathcal{G} \times \mathcal{G}$ and will therefore not be controllable.

This chapter addresses the following question: under what conditions does the component-wise controllability of a control-coupled system imply that the whole system is controllable?

The answer lies hidden above. As was noted there, if $X_i = Y_i$ for all $i = 1, \dots, m$, then system (8.1.1) is not controllable. The condition $X_i = Y_i$ can be written as $Y_i = \mathbb{I}_{\mathfrak{g}}(X_i)$, where $\mathbb{I}_{\mathfrak{g}}$ is the identity map, or, to be pedantic, the identity *automorphism* of \mathfrak{g} .

It is similarly true that system (8.1.1) is not controllable if, for all $i = 1, \dots, m$, $Y_i = \theta(X_i)$ for *any* automorphism of \mathfrak{g} . For, in this case, the reachable sets of (8.1.1) will be cosets of the subgroup of $\mathcal{G} \times \mathcal{G}$ whose Lie algebra is the *graph* of θ ,

$$\text{graph } \theta := \left\{ \begin{pmatrix} X \\ \theta X \end{pmatrix} \mid X \in \mathfrak{g} \right\}$$

The main result in this chapter states that so long as \mathcal{G} is a simple group, and so long as system (8.1.1) is component-wise controllable, the existence of an automorphism θ of \mathfrak{g} satisfying $Y_i = \theta(X_i)$ for all $i = 1, \dots, m$ is the *only* way that (8.1.1) can fail to be controllable.

Theorem 8.1.1 (The Automorphism Condition, Version I). *In the notation of (8.1.1), suppose that \mathcal{G} is a simple group and that (8.1.1) is component-wise controllable. Then, (8.1.1) is controllable if and only if there is no automorphism of \mathfrak{g} satisfying $\theta X_i = Y_i$ for all $i = 1, \dots, m$.*

The rest of this chapter is committed to proving Theorem 8.1.1. The chapter is organized as follows: the requisite background material from Lie theory and control theory is presented in Section 8.2. The proof of the Automorphism Condition comes in Section 8.3. In Section 8.4, the Automorphism Condition is

applied to coupled systems on the classical simple real Lie groups $SO(n)$ and $SU(n)$. Section 8.5 explores generalizations of The Automorphism Condition.

8.2 BACKGROUND MATERIAL

The proof of the Automorphism Condition requires some definitions from Lie theory and some basic results from control theory. These are presented here, mostly without proof.

8.2.1 DEFINITIONS FROM LIE THEORY

This section requires some familiarity with elementary facts about Lie groups and Lie algebras. For a reference, see [25]. All Lie algebras appearing here will be real and finite-dimensional. Throughout this chapter, the Lie algebra of a Lie group will be thought of interchangeably as the Lie algebra of left-invariant vector fields on the group, and as the tangent space to the identity of the group. If \mathcal{G} is a Lie group, its Lie algebra will be denoted by $\text{Lie}(\mathcal{G})$. The identity in \mathcal{G} will be written as e .

If \mathfrak{h} is the Lie algebra of the Lie group \mathcal{H} and \mathfrak{g} the Lie algebra of the Lie group \mathcal{G} , then the Lie algebra of the direct product $\mathcal{G} \times \mathcal{H}$ will be identified with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$, whose Lie bracket is just the component-wise bracket : $[(A, B), (C, D)] = ([A, C], [B, D])$ for $(A, B), (C, D) \in \mathfrak{g} \oplus \mathfrak{h}$. As was mentioned in the introduction to this chapter, elements of a direct sum such as $\mathfrak{g} \oplus \mathfrak{h}$ will sometimes be written as a column,

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

for $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$, in which case the direct-sum Lie bracket will be written as

$$\left[\begin{pmatrix} A \\ B \end{pmatrix}, \begin{pmatrix} C \\ D \end{pmatrix} \right] = \begin{pmatrix} [A, C] \\ [B, D] \end{pmatrix}$$

A linear map $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ between Lie algebras \mathfrak{g} and \mathfrak{h} is a *homomorphism* if $\theta([X, Y]) = [\theta X, \theta Y]$ for all $X, Y \in \mathfrak{g}$. A homomorphism that is also a linear isomorphism will be called a *Lie algebra isomorphism*. A Lie algebra isomorphism from a Lie algebra \mathfrak{g} to itself is an *automorphism*. The set of automorphisms of \mathfrak{g} will be denoted by $\text{Aut}(\mathfrak{g})$.

The projections $\pi_1 : \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{g}$ and $\pi_2 : \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{h}$ from a direct sum to its constituents are both homomorphisms. The restriction of a homomorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ to a subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a homomorphism from \mathfrak{k} to \mathfrak{h} and will be denoted by $\theta|_{\mathfrak{k}}$.

For any subset $S \subset \mathfrak{g}$, there is a smallest subalgebra of \mathfrak{g} that contains S , namely the intersection of all subalgebras containing S . This subalgebra will be referred to as the subalgebra generated by S and will be denoted by $\langle S \rangle$.

Proposition 8.2.1. *If $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism, then $\theta\langle S \rangle = \langle \theta(S) \rangle$*

Definition 8.2.2. *A Lie group is **simple** if it is connected and if it has no proper connected normal subgroups. A Lie algebra is **simple** if it contains no proper ideals.*

Proposition 8.2.3. *The Lie algebra of a simple Lie group is simple.*

Remark 8.2.4. The matrix Lie algebras $\mathfrak{su}(n)$ is simple for all $n \geq 1$. $\mathfrak{so}(n)$ is simple for all $n \neq 4$.

Lemma 8.2.5. *Let \mathfrak{g} and \mathfrak{h} be Lie algebras with \mathfrak{g} simple, and let $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ be a homomorphism. If $\theta \neq 0$ then θ is injective. Thus, if θ is surjective, it is a Lie algebra isomorphism.*

Proof. The kernel of θ is an ideal in \mathfrak{g} and is therefore either 0 or \mathfrak{g} . If $\theta \neq 0$ then $\ker \theta = 0$. □

8.2.2 RESULTS FROM CONTROL THEORY

System (8.1.1) is an example of a driftless left-invariant bilinear control system, which in general have the following form,

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)) \quad (8.2.1)$$

where the X_i are left-invariant vector fields on a Lie group \mathcal{G} , u_i controls and γ a differentiable curve on \mathcal{G} .

The system (8.2.1) is *controllable* if for any $\Lambda \in \mathcal{G}$ there exist controls $u_i : \mathbb{R} \rightarrow \mathbb{R}$ and a differentiable curve $\gamma : \mathbb{R} \rightarrow \mathcal{G}$ satisfying Equation (8.1.1) and the constraints $\gamma(0) = I_{\mathcal{G}}$, $\gamma(1) = \Lambda$. The *reachable set* (from the identity) of system (8.2.1) is the set of $\Lambda \in \mathcal{G}$ for which such controls and such a curve exist.

This paper's main result rests on the following well-known controllability criterion for driftless bilinear control systems on Lie groups; it appears as Theorem 7.1 in [22].

Theorem 8.2.6. *The reachable set (from the identity) of system (8.2.1) is the connected subgroup of \mathcal{G} whose Lie algebra is $\langle X_1, \dots, X_m \rangle$. Therefore, if \mathcal{G} is connected, system (8.2.1) is controllable if and only if $\langle X_1, \dots, X_m \rangle = \text{Lie}(\mathcal{G})$.*

Applied to system (8.1.1), Theorem 8.2.6 yields the following criterion.

Corollary 8.2.7. *System (8.1.1) is controllable if and only if*

$$\left\langle \left\{ \begin{pmatrix} X_i \\ Y_i \end{pmatrix} : i = 1, \dots, m \right\} \right\rangle = \mathfrak{g} \oplus \mathfrak{g}$$

On the other hand, and again by Theorem 8.2.6, system (8.1.1) is *component-wise* controllable if and only if $\langle X_i : i = 1, \dots, m \rangle = \langle Y_i : i = 1, \dots, m \rangle = \mathfrak{g}$. The central question of this chapter, then, is this: given two collections of elements of \mathfrak{g} , $\{X_i : i = 1, \dots, m\}$ and $\{Y_i : i = 1, \dots, m\}$, such

that $\langle X_i : i = 1, \dots, m \rangle = \langle Y_i : i = 1, \dots, m \rangle = \mathfrak{g}$, when is it true that

$$\left\langle \left\{ \begin{pmatrix} X_i \\ Y_i \end{pmatrix} : i = 1, \dots, m \right\} \right\rangle = \mathfrak{g} \oplus \mathfrak{g}$$

8.2.3 GRAPHS OF HOMOMORPHISMS

The proof of the Automorphism Condition invokes the notion of a graph of a homomorphism.

Definition 8.2.8. Let $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ be a homomorphism between the Lie algebras \mathfrak{g} and \mathfrak{h} . The **graph** of θ is the subset

$$\left\{ \begin{pmatrix} X \\ \theta X \end{pmatrix} : X \in \mathfrak{g} \right\} \subset \mathfrak{g} \oplus \mathfrak{h}$$

The graph of θ will be denoted by $\text{graph } \theta$.

Lemma 8.2.9. The graph of a homomorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ is a subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$ satisfying $\pi_1(\text{graph } \theta) = \mathfrak{g}$ and $\dim \text{graph } \theta = \dim \mathfrak{g}$ (where $\pi_1 : \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{g}$ is the projection map).

Proof. The linearity of θ implies that $\text{graph } \theta$ is a linear subspace of $\mathfrak{g} \oplus \mathfrak{h}$. It is a subalgebra because, for $X, Y \in \mathfrak{g}$,

$$\begin{aligned} & \left[\begin{pmatrix} X \\ \theta X \end{pmatrix}, \begin{pmatrix} Y \\ \theta Y \end{pmatrix} \right] \\ &= \begin{pmatrix} [X, Y] \\ [\theta X, \theta Y] \end{pmatrix} \\ &= \begin{pmatrix} [X, Y] \\ \theta[X, Y] \end{pmatrix} \end{aligned}$$

which is again in $\text{graph } \theta$. If $X \in \mathfrak{g}$, then $\pi_1(X, \theta X) = X$. Thus, $\pi_1(\text{graph } \theta) = \mathfrak{g}$. Finally, the map from \mathfrak{g} to $\text{graph } \theta$ taking $X \in \mathfrak{g}$ to $(X, \theta X) \in \text{graph } \theta$ is a Lie algebra isomorphism. It follows that $\dim \text{graph } \theta = \dim \mathfrak{g}$. □

Remark 8.2.10. The second half of the lemma implies that the graph of a homomorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ is necessarily a *proper* subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$. ■

The converse of Lemma 8.2.9 is true as well; it plays an important role in the proof of the Automorphism Condition.

Lemma 8.2.11. *A subalgebra $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{h}$ is the graph of a homomorphism $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ (i.e., $\mathfrak{k} = \text{graph } \theta$) if and only if $\pi_1(\mathfrak{k}) = \mathfrak{g}$ and $\dim \mathfrak{k} = \dim \mathfrak{g}$.*

Proof. The necessity of the two conditions in the lemma was established in Lemma 8.2.9. Conversely, the conditions $\pi_1(\mathfrak{k}) = \mathfrak{g}$ and $\dim \mathfrak{k} = \dim \mathfrak{g}$ imply that the restriction $\pi_1|_{\mathfrak{k}}$ of π_1 to \mathfrak{k} is a Lie algebra isomorphism from \mathfrak{k} to \mathfrak{g} . It therefore has an inverse $(\pi_1|_{\mathfrak{k}})^{-1} : \mathfrak{g} \rightarrow \mathfrak{k}$. If $i_{\mathfrak{k}} : \mathfrak{k} \hookrightarrow \mathfrak{g} \oplus \mathfrak{h}$ denotes the inclusion homomorphism of \mathfrak{k} , then $\theta = \pi_2 \circ i_{\mathfrak{k}} \circ (\pi_1|_{\mathfrak{k}})^{-1} : \mathfrak{g} \rightarrow \mathfrak{k} \rightarrow \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{h}$ is a homomorphism and $\text{graph } \theta = \mathfrak{k}$. □

Corollary 8.2.12. *A subalgebra $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ is the graph of an automorphism $\theta \in \text{Aut}(\mathfrak{g})$ if and only if $\dim \mathfrak{k} = \dim \mathfrak{g}$ and $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$.*

Corollary 8.2.12 establishes a bijective correspondence between automorphisms of \mathfrak{g} and subalgebras $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$ and $\dim \mathfrak{k} = \dim \mathfrak{g}$. This correspondence will play an important role in the proof of the Automorphism Condition.

The proof of necessity in the Automorphism Condition invokes the following lemma, the last of this section.

Lemma 8.2.13. *Let $X_1, \dots, X_m \in \mathfrak{g}$ and let $\theta : \mathfrak{g} \rightarrow \mathfrak{h}$ be a homomorphism. Then, $\langle \{(X_i, \theta X_i)\} \rangle \subset \text{graph } \theta$, with equality if and only if $\langle \{X_i\} \rangle = \mathfrak{g}$.*

8.3 A PROOF OF THE AUTOMORPHISM CONDITION

The Automorphism Condition has an equivalent formulation that is, at least notationally, slightly easier to prove; it eliminates the need to refer to the elements $(X_i, Y_i) \in \mathfrak{g} \oplus \mathfrak{g}$.

Theorem 8.3.1 (The Automorphism Condition, Version II). *Let \mathfrak{g} be a simple Lie algebra. Let $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ be a subalgebra such that $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$. Then, \mathfrak{k} is proper if and only if $\mathfrak{k} = \text{graph } \theta$ for some $\theta \in \text{Aut}(\mathfrak{g})$.*

Remark 8.3.2. Version I of the Automorphism Condition (Theorem 8.1.1) can be recovered from Version II (Theorem 8.3.1) by setting $\mathfrak{k} = \langle \{(X_i, Y_i)\} \rangle$. On the other hand, Version I implies Version II: given a subalgebra $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ let $\langle \{(X_i, Y_i)\} \rangle$ be a basis for \mathfrak{k} and apply Theorem 8.1.1. ■

Remark 8.3.3. The assumption that \mathfrak{g} is simple in Theorem 8.3.1 is indispensable. For example, if \mathfrak{g} is semisimple, but not simple, it is always possible to find *proper* subalgebras $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$ and $\mathfrak{k} \neq \text{graph } \theta$ for any $\theta \in \text{Aut}(\mathfrak{g})$. This is discussed in Section 8.5. ■

If $\mathfrak{k} = \text{graph } \theta$ for some $\theta \in \text{Aut}(\mathfrak{g})$, then \mathfrak{k} is necessarily a proper subalgebra. The remainder of this section is devoted to proving the converse statement, that *every* proper subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$ is necessarily the graph of some automorphism. Corollary 8.2.12 provides a means for doing this.

Corollary 8.2.12 says that a subalgebra $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$ is the graph of an automorphism if and only if $\dim \mathfrak{k} = \dim \mathfrak{g}$. Thus, Theorem 8.3.1 stands if and only if the only subalgebra \mathfrak{k} of $\mathfrak{g} \oplus \mathfrak{g}$ satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$ and $\dim \mathfrak{k} > \dim \mathfrak{g}$ is $\mathfrak{g} \oplus \mathfrak{g}$ itself. That is, if \mathfrak{k} is a *proper* subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$, then it must satisfy $\dim \mathfrak{k} = \dim \mathfrak{g}$ (and therefore be a graph). By this reasoning the following lemma and its proof complete the proof of Theorem 8.3.1.

Lemma 8.3.4. *Let \mathfrak{g} be a simple Lie algebra and $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ a subalgebra satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$. If $\dim \mathfrak{k} > \dim \mathfrak{g}$, then $\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g}$.*

Remark 8.3.5. Lemma 8.3.4 says that there are no subalgebras \mathfrak{k} of $\mathfrak{g} \oplus \mathfrak{g}$ satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$ and having dimension $\dim \mathfrak{g} < \dim \mathfrak{k} < 2 \dim \mathfrak{g}$. ■

The proof of Lemma 8.3.4 requires a small number of intermediate lemmas. But first, given a subalgebra $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$, define an associated subalgebra $\mathfrak{s}_{\mathfrak{k},1} \subset \mathfrak{g}$ by

$$\mathfrak{s}_{\mathfrak{k},1} = \{X \in \mathfrak{g} : (X, 0) \in \mathfrak{k}\}$$

and define $\mathfrak{s}_{\mathfrak{k},2} \subset \mathfrak{g}$ analogously ($\mathfrak{s}_{\mathfrak{k},2} = \{X \in \mathfrak{g} : (0, X) \in \mathfrak{k}\}$).

Proposition 8.3.6. $\mathfrak{s}_{\mathfrak{k},1} = \pi_1(\mathfrak{k} \cap \ker \pi_2) = \pi_1|_{\mathfrak{k}}(\ker \pi_2|_{\mathfrak{k}})$. An analogous result holds for $\mathfrak{s}_{\mathfrak{k},2}$.

Proposition 8.3.7. $\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g}$ if and only if $\mathfrak{s}_{\mathfrak{k},1} = \mathfrak{s}_{\mathfrak{k},2} = \mathfrak{g}$.

Proposition 8.3.7 suggests an approach to proving Lemma 8.3.4, namely by showing that under the hypotheses of the lemma, $\mathfrak{s}_{\mathfrak{k},1} = \mathfrak{s}_{\mathfrak{k},2} = \mathfrak{g}$. The next two lemmas start this pursuit by showing that the \mathfrak{s}_i are ideals in \mathfrak{g} .

Lemma 8.3.8. Let \mathfrak{g} and \mathfrak{h} be Lie algebras, $\mathfrak{j} \subset \mathfrak{g}$ an ideal, and let $\pi : \mathfrak{g} \rightarrow \mathfrak{h}$ be a surjective homomorphism. The image of \mathfrak{j} under π is an ideal in \mathfrak{h} .

Proof. Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Because π is surjective, $Y = \pi(Z)$ for some $Z \in \mathfrak{g}$. Thus, $[Y, \pi(X)] = [\pi(Z), \pi(X)] = \pi[Z, X] \in \text{Im } \pi$. □

Lemma 8.3.9. Let \mathfrak{g} be a Lie algebra and $\mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ a subalgebra satisfying $\pi_1(\mathfrak{k}) = \pi_2(\mathfrak{k}) = \mathfrak{g}$. Then, the subsets $\mathfrak{s}_{\mathfrak{k},1}$ and $\mathfrak{s}_{\mathfrak{k},2}$ are ideals in \mathfrak{g} .

Proof. $\mathfrak{k} \cap \ker \pi_2 = \ker(\pi_2|_{\mathfrak{k}})$ is an ideal in \mathfrak{k} . By Lemma 8.3.6, $\mathfrak{s}_{\mathfrak{k},1}$ is the image of this ideal under the homomorphism $\pi_1|_{\mathfrak{k}}$, which is surjective by hypothesis. Lemma 8.3.8 implies that $\mathfrak{s}_{\mathfrak{k},1}$ is an ideal. Applied to $\mathfrak{k} \cap \ker \pi_1$ the same argument implies that $\mathfrak{s}_{\mathfrak{k},2}$ is an ideal too. □

Corollary 8.3.10. If \mathfrak{g} is a simple Lie algebra, then, under the assumptions of Lemma 8.3.9, each $\mathfrak{s}_{\mathfrak{k},i}$ is either $\{0\}$ or \mathfrak{g} .

Proof. (Proof of Lemma 8.3.4) The conditions $\pi_2(\mathfrak{k}) = \mathfrak{g}$ and $\dim \mathfrak{k} > \dim \mathfrak{g}$ imply that $\ker(\pi_2|_{\mathfrak{k}}) \neq \{0\}$. Since $\ker \pi_1 \cap \ker \pi_2 = \{0\}$, it follows that $\mathfrak{s}_{\mathfrak{k},1} = \pi_1(\ker(\pi_2|_{\mathfrak{k}}))$ is not equal to $\{0\}$. Corollary 8.3.10 then implies that $\mathfrak{s}_{\mathfrak{k},1} = \mathfrak{g}$. The equality $\mathfrak{s}_{\mathfrak{k},2} = \mathfrak{g}$ follows similarly. By Claim 8.3.7, $\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g}$. This completes the proof of the Automorphism Condition. \square

8.4 COROLLARIES

This section discusses some applications of the Automorphism Condition.

The automorphism group of a Lie algebra \mathfrak{g} acts on the m -fold sum $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ according to $(X_1, \dots, X_m) \mapsto (\theta X_1, \dots, \theta X_m)$ for $\theta \in \text{Aut}(\mathfrak{g})$. This action will henceforth be referred to as the *diagonal* action of $\text{Aut}(\mathfrak{g})$ and will be denoted by \mathcal{A}_D . If $(X_1, \dots, X_m) \in \bigoplus_{i=1}^m \mathfrak{g}$, the orbit of \mathcal{A}_D passing through (X_1, \dots, X_m) will be denoted by $\mathcal{O}_{(X_1, \dots, X_m)}$.

In terms of the diagonal action of $\text{Aut}(\mathfrak{g})$, the Automorphism Condition says that component-wise controllability of system (8.1.1) is sufficient for its full controllability so long as the tuples (X_1, \dots, X_m) and (Y_1, \dots, Y_m) are *not* in the same orbit of the diagonal action. Using the Automorphism Condition therefore requires some means of determining whether two points in $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ lie in the same orbit of the diagonal action.

In the classical matrix algebras $\mathfrak{so}(n)$ and $\mathfrak{su}(n)$ such means are available. For $n \neq 1, 2, 4$, every automorphism of $\mathfrak{so}(n)$ is eigenvalue-preserving. Every automorphism of $\mathfrak{su}(n)$ either preserves the eigenvalues of its argument or complex-conjugates them. These facts are discussed in Appendix 8.6 (see Corollaries 8.6.6 and 8.6.7). They yield the following corollaries.

Corollary 8.4.1. *Suppose $\mathcal{G} = \text{SO}(n)$ in system (8.1.1) (with $n \neq 1, 2, 4$; see Remark 8.4.3 below). Then, system (8.1.1) is controllable if it is component-wise controllable and if for some $i \in \{1, \dots, m\}$ the set of eigenvalues of X_i does not equal the set of eigenvalues of Y_i .*

Corollary 8.4.2. Suppose $\mathcal{G} = SU(n)$ in system (8.1.1). Then, system (8.1.1) is controllable if it is component-wise controllable and if for some $i \in \{1, \dots, m\}$ the set of eigenvalues of X_i equals neither the set of eigenvalues of Y_i nor the set of complex-conjugates of the eigenvalues of Y_i .

Remark 8.4.3. The restriction $n \neq 1, 2, 4$ in Corollary 8.4.1 is made necessary by the fact that, for those values of n , $SO(n)$ is not a simple group. ■

The conditions in Corollaries 8.4.1 and 8.4.2 are not necessary for the controllability of system (8.1.1). Points (X_1, \dots, X_m) and (Y_1, \dots, Y_m) in $\mathfrak{so}(n) \oplus \dots \oplus \mathfrak{so}(n)$ (and $\mathfrak{su}(n) \oplus \dots \oplus \mathfrak{su}(n)$) may lie in *distinct* orbits of \mathcal{A}_D but still have the property that their i^{th} components X_i and Y_i have the same eigenvalues for $i = 1, \dots, m$. The coupled systems (8.1.1) defined by such points fail to satisfy the conditions of Corollaries 8.4.1 and 8.4.2 but are nevertheless controllable. The following elements of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ provide an example of this.

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \quad \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \end{pmatrix}$$

The eigenvalues of X_1 and Y_1 are the same, as are those of X_2 and Y_2 . Indeed, $X_1 = Y_1$, and $Y_2 = \theta X_2 \theta^T$ for

$$\theta = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

However, there is no automorphism mapping X_1 to Y_1 and X_2 to Y_2 . This follows from two facts (the first of which is explained in Corollary 8.6.4 in Appendix

8.6): that every automorphism of $\mathfrak{so}(3)$ is of the form $X \mapsto \theta X \theta^T$ for some $\theta \in SO(3)$, and that the bilinear form $(X, Y) \mapsto \text{Tr}(XY^T)$ on $\mathfrak{so}(3)$ is preserved by such automorphisms. Thus, the existence of an automorphism mapping X_1 to Y_1 and X_2 to Y_2 would imply the equality $\text{Tr}(X_1 X_2^T) = \text{Tr}(Y_1 Y_2^T)$. However, $\text{Tr}(X_1 X_2^T) = 0$ while $\text{Tr}(Y_1 Y_2^T) \neq 0$. Thus, no such automorphism exists.

In short, the eigenvalues of the components of points in $\mathfrak{so}(n) \oplus \cdots \oplus \mathfrak{so}(n)$ (or $\mathfrak{su}(n) \oplus \cdots \oplus \mathfrak{su}(n)$) do not fully characterize the orbits of the diagonal action.

In general, given two points (X_1, \dots, X_m) and (Y_1, \dots, Y_m) in $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$, it may be that for each $i = 1, \dots, m$ there exists an automorphism $\theta_i \in \text{Aut}(\mathfrak{g})$ satisfying $\theta_i X_i = Y_i$, but there exists *no single automorphism* $\theta \in \text{Aut}(\mathfrak{g})$ satisfying $\theta X_i = Y_i$ for each $i = 1, \dots, m$. This was the case in the preceding example and explains the failure of the conditions in Corollaries **8.4.1** and **8.4.2** to be necessary ones.

There is at least one situation in which the orbits of the diagonal action can be described completely, namely when $\mathfrak{g} = \mathfrak{so}(3)$ and $m = 2$. Corollary **8.4.7** below states that the orbits of the diagonal action of $\text{Aut}(\mathfrak{so}(3))$ on $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ are the level sets of a certain function $F_{\mathfrak{so}(3),2}$ from $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ to Sym_2 , the set of real, symmetric 2×2 matrices. As a result, when $\mathcal{G} = SO(3)$ and $m = 2$ in **(8.1.1)**, the problem of determining the controllability of system **(8.1.1)** reduces to computing $F_{\mathfrak{so}(3),2}$. The system **(8.1.1)** is controllable if and only if, in the terms of equation **(8.1.1)**, $F_{\mathfrak{so}(3),2}(X_1, X_2) = F_{\mathfrak{so}(3),2}(Y_1, Y_2)$.

Corollary **8.4.7** is prefaced by a slight but convenient change of perspective. The diagonal action of $\text{Aut}(\mathfrak{so}(3))$ on $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is equivalent to the diagonal action of $SO(3)$ on $\mathbb{R}^3 \oplus \mathbb{R}^3$ (the action taking $(\theta, (X, Y)) \in SO(3) \times \mathbb{R}^3 \oplus \mathbb{R}^3$ to $(\theta X, \theta Y)$). This is explained in detail below. The advantage of this equivalence is that the diagonal action of $SO(3)$ on $\mathbb{R}^3 \oplus \mathbb{R}^3$ is, at least notationally, easier to study than the diagonal action on $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. For this reason, Corollary **8.4.7** is reached by initially studying the action on $\mathbb{R}^3 \oplus \mathbb{R}^3$.

Lemma 8.4.4. *Let $(X, Y), (X', Y')$ be elements of $\mathbb{R}^3 \oplus \mathbb{R}^3$. (X, Y) and (X', Y') are in the same orbit of the action of $SO(3)$ on $\mathbb{R}^3 \oplus \mathbb{R}^3$ if and only if the following*

equalities hold : $\|X\| = \|X'\|$, $\|Y\| = \|Y'\|$ and $\langle X, Y \rangle = \langle X', Y' \rangle$ (where \langle, \rangle denotes the standard inner product on \mathbb{R}^3 and $\|\cdot\|$ its associated norm). Equivalently, the orbits of the action of $SO(3)$ on $\mathbb{R}^2 \oplus \mathbb{R}^3$ are the level sets of the function $F_{\mathfrak{so}(3),2} : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \rightarrow \text{Sym}_2$ given by

$$(X, Y) \mapsto \begin{pmatrix} \|X\|^2 & \langle X, Y \rangle \\ \langle X, Y \rangle & \|Y\|^2 \end{pmatrix}$$

Remark 8.4.5. The choice of Sym_2 , rather than \mathbb{R}^3 , as the codomain of $F_{\mathfrak{so}(3),2}$ will make it easier to define a generalization of $F_{\mathfrak{so}(3),2}$. This is done in Remark 8.4.9 below and in Section 8.5.3. The subscript 2 on $F_{\mathfrak{so}(3),2}$ refers to the fact that $F_{\mathfrak{so}(3),2}$ is defined on the 2-fold direct sum of $\mathfrak{so}(3)$ with itself. ■

The proof of Lemma 8.4.4 requires a lemma of its own. Let $\mathbb{R}^2 \subset \mathbb{R}^3$ denote the subspace of \mathbb{R}^3 whose third coordinate is zero, and let $O(2) = \{\theta \in SO(3) : \theta(\mathbb{R}^2) = \mathbb{R}^2\}$ be the subgroup of $SO(3)$ that preserves \mathbb{R}^2 . The next lemma establishes the result in Lemma 8.4.4 in the event that X, X', Y and Y' all lie in \mathbb{R}^2 .

The proof of Lemma 8.4.6 works by rotating X into X' by some $\theta \in SO(2)$, so that $\theta X = X'$. This is made possible by the equality $\|X\| = \|X'\|$. The remaining equalities, $\|Y\| = \|Y'\|$ and $\langle X, Y \rangle = \langle X', Y' \rangle$ show that θY is one of $\pm Y'$. If $\theta Y = Y'$, the proof is complete. In the event that $\theta Y = -Y'$, θ can be composed with a reflection from $O(2)$. The details of this argument are below.

Lemma 8.4.6. *Let $(X, Y), (X', Y')$ be elements of $\mathbb{R}^2 \oplus \mathbb{R}^2 \subset \mathbb{R}^3 \oplus \mathbb{R}^3$. (X, Y) and (X', Y') are in the same orbit of the action of $O(2)$ on $\mathbb{R}^2 \oplus \mathbb{R}^2$ if and only if the following equalities hold : $\|X\| = \|X'\|$, $\|Y\| = \|Y'\|$ and $\langle X, Y \rangle = \langle X', Y' \rangle$ (here the inner product and norm are the ones induced on \mathbb{R}^2 by those on \mathbb{R}^3).*

Proof. If (X, Y) and (X', Y') are in the same orbit, then $X' = \theta X$ and $Y' = \theta Y$ for some $\theta \in O(2)$, in which case the equalities in the lemma must hold. Conversely, assume that the equalities are true. If $X' = 0$, then the equality $\|X\| = \|X'\|$ implies that $X = 0$ as well. The equality $\|Y\| = \|Y'\|$ then implies that there exists a $\theta \in O(2)$ such that $(0, \theta Y) = (0, \theta Y')$. This completes the proof in the event that $X' = 0$.

If $X' \neq 0$, let $n_{X'}$ be a unit-norm element orthogonal to X' . Then, Y' is equal to one of the following elements,

$$\frac{\langle X', Y' \rangle}{\|X'\|^2} X' \pm \left(\sqrt{\|Y'\|^2 - \left(\frac{\langle X', Y' \rangle}{\|X'\|^2} \right)^2} \right) n_{X'} \quad (8.4.1)$$

The equality $\|X\| = \|X'\|$ implies the existence of a $\theta \in SO(2)$ satisfying $X' = \theta X$. θY is equal to one of the following elements,

$$\frac{\langle X', \theta Y \rangle}{\|X'\|^2} X' \pm \left(\sqrt{\|\theta Y\|^2 - \left(\frac{\langle X', \theta Y \rangle}{\|X'\|^2} \right)^2} \right) n_{X'} \quad (8.4.2)$$

The remaining equalities, $\|Y\| = \|Y'\|$, $\langle X, Y \rangle = \langle X', Y' \rangle$ imply that each element in Equation 8.4.2 is equal to the corresponding element in Equation 8.4.1. Thus, Y' and θY are therefore both elements of the two element set

$$\left\{ \frac{\langle X', Y' \rangle}{\|X'\|^2} X' \pm \left(\sqrt{\|Y'\|^2 - \left(\frac{\langle X', Y' \rangle}{\|X'\|^2} \right)^2} \right) n_{X'} \right\}$$

If $Y = Y'$, then $\theta \cdot (X, Y) = (X', Y')$. If $Y \neq Y'$, then $(r_{X'} \theta) \cdot (X, Y) = (X', Y')$ where $r_{X'} \in O(2)$ is the orthogonal reflection about X' . In either case, (X, Y) and (X', Y') are in the same orbit of the action of $O(2)$. \square

The proof of Lemma 8.4.4 now proceeds by simultaneously rotating X and Y into \mathbb{R}^2 , then doing the same for X' and Y' , and then finally applying Lemma 8.4.6.

Proof. (Proof of Lemma 8.4.4) If (X, Y) and (X', Y') are in the same orbit, then $X' = \theta X$ and $Y' = \theta Y$ for some $\theta \in SO(3)$, in which case the equalities in the lemma must hold. For the converse, note that, for any pairs (X, Y) and (X', Y') , there exist $\theta_1, \theta_2 \in SO(3)$ which are such that $\theta_1 \cdot (X, Y)$ and $\theta_2 \cdot (X', Y')$ are both in $\mathbb{R}^2 \oplus \mathbb{R}^2 \subset \mathbb{R}^3 \oplus \mathbb{R}^3$. Lemma 8.4.6 then implies that there exists a $\theta_3 \in O(2)$ satisfying $(\theta_3 \theta_2) \cdot (X, Y) = \theta_1(X', Y')$, or $(\theta_1^{-1} \theta_3 \theta_2) \cdot (X, Y) = (X', Y')$. Thus, (X, Y) and (X', Y') are in the same orbit of the action of $SO(3)$. \square

The task of relating Lemma 8.4.4 to the diagonal action of $\text{Aut}(\mathfrak{so}(3))$ on $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ remains. This is made possible by the fact that $\mathfrak{so}(3)$ and \mathbb{R}^3 are isomorphic as Lie algebras when the latter is thought of as a Lie algebra whose bracket is the cross-product. For example, the linear map $L : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ defined by

$$e_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad e_3 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $e_i \in \mathbb{R}^3$ are the standard basis vectors of \mathbb{R}^3 , is a Lie algebra isomorphism.

$\text{Aut}(\mathfrak{so}(3))$ is isomorphic to $SO(3)$, which acts on $\mathfrak{so}(3)$ by conjugation. This was observed above (and is explained in Corollary 8.6.4 in Appendix 8.6). It follows that $\text{Aut}(\mathbb{R}^3)$ (the automorphism group of \mathbb{R}^3 , thought of as a Lie algebra) is also isomorphic to $SO(3)$. In fact, $\text{Aut}(\mathbb{R}^3) = SO(3)$.

The isomorphism L intertwines the action of $SO(3)$ on \mathbb{R}^3 with the action of $SO(3)$ on $\mathfrak{so}(3) : L(\theta X) = \theta L(X)\theta^T$ for all $X \in \mathbb{R}^3$ and $\theta \in SO(3)$. L can therefore be used to identify the diagonal action of $SO(3)$ on $\mathbb{R}^3 \oplus \mathbb{R}^3$ with the diagonal action on $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

In addition to being a Lie algebra isomorphism, L is an isometry between \mathbb{R}^3 , with its standard inner product, and $\mathfrak{so}(3)$, whose inner product is the negative of its Killing form. Putting these facts together yields the following corollary, which is simply Lemma 8.4.4 translated into the language of $\mathfrak{so}(3)$.

Corollary 8.4.7. *When $\mathcal{G} = SO(3)$ and $m = 2$, the system (8.1.1) is controllable if and only if it is component-wise controllable and $F_{\mathfrak{so}(3),2}(X_1, X_2) = F_{\mathfrak{so}(3),2}(Y_1, Y_2)$ where $F_{\mathfrak{so}(3),2} : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \rightarrow \text{Sym}_2$ is given by*

$$(X, Y) \mapsto \begin{pmatrix} B(X, X) & B(X, Y) \\ B(X, Y) & B(Y, Y) \end{pmatrix}$$

where B denotes the Killing form on $\mathfrak{so}(3)$.

Because $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic, and isometric with respect to their Killing forms, Corollary 8.4.7 automatically provides a necessary and sufficient condition for the controllability of coupled systems on $SU(2) \times SU(2)$.

Corollary 8.4.8. *When $\mathcal{G} = SU(2)$ and $m = 2$, the system (8.1.1) is controllable if and only if it is component-wise controllable and $F_{\mathfrak{so}(3),2}(X_1, X_2) = F_{\mathfrak{so}(3),2}(Y_1, Y_2)$ where $F_{\mathfrak{so}(3),2} : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \rightarrow \text{Sym}_2$ is given by*

$$(X, Y) \mapsto \begin{pmatrix} B(X, X) & B(X, Y) \\ B(X, Y) & B(Y, Y) \end{pmatrix}$$

where B denotes the Killing form on $\mathfrak{su}(2)$.

Remark 8.4.9. $F_{\mathfrak{so}(3),2}$ can be generalized to a function $F_{\mathfrak{g},m} : \mathfrak{g} \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g} \rightarrow \text{Sym}_m$ defined on the m -fold direct sum of any Lie algebra \mathfrak{g} with itself: $F_{\mathfrak{g},m}(X_1, \dots, X_m)$ is the symmetric $m \times m$ matrix whose $(i, j)^{\text{th}}$ entry is $B(X_i, X_j)$ where B is the Killing form of \mathfrak{g} . The matter of whether or not $F_{\mathfrak{g},m}$ serves the same purpose as $F_{\mathfrak{so}(3),2}$, whether the orbits of \mathcal{A}_D in $\mathfrak{g} \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ are the level sets of $F_{\mathfrak{g},m}$, is addressed in Section 8.5.3. For now it is observed that when $\dim \mathfrak{g} \geq 2$ and B is definite and nondegenerate (as it is when \mathfrak{g} is compact, for example), $F_{\mathfrak{g},2}$ has a regular value whose preimage is not empty.

Lemma 8.4.10. *Let \mathfrak{g} be a Lie algebra with $\dim \mathfrak{g} \geq 2$ and suppose that its Killing form is definite and nondegenerate. Let $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ be such that $X \neq 0$, $Y \neq 0$ and $B(X, X) \neq B(Y, Y)$ (in particular, $X \neq Y$). Then, $F_{\mathfrak{g},2}(X, Y) \in \text{Sym}_2$ is a regular value of $F_{\mathfrak{g},2}$.*

Lemma 8.4.10 needs a lemma of its own. Lemma 8.4.11 makes the usual identification of the tangent space to a point in a vector space with the vector space itself.

Lemma 8.4.11. *Let \mathfrak{g} , X and Y be as they are in Lemma 8.4.10. Then, $(dF_{\mathfrak{g},2})_{(X,Y)} : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \text{Sym}_2$, the differential of $F_{\mathfrak{g},2}$ at $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ is surjective.*

Proof. $(dF_{g,2})_{(X,Y)}$ is given by

$$(V, W) \mapsto \begin{pmatrix} 2B(V, X) & B(V, Y) + B(X, W) \\ B(V, Y) + B(X, W) & 2B(W, Y) \end{pmatrix}$$

Using the nongeneracy of B , let V be a nonzero element of $Y^\perp - X^\perp$, where Y^\perp and X^\perp respectively denote the orthogonal complement of Y and X with respect to B ($Y^\perp - X^\perp$ is nonzero because of that assumption that $X \neq Y$). Suppose that $B(V, X) = 1/2$ (this is possible by the assumption that $V \in (X^\perp)^\circ$). Then,

$$(V, o) \mapsto \begin{pmatrix} 1 & o \\ o & o \end{pmatrix} \quad (8.4.3)$$

Similarly, for any $W \in X^\perp - Y^\perp$ with $B(W, Y) = 1/2$,

$$(o, W) \mapsto \begin{pmatrix} o & o \\ o & 1 \end{pmatrix} \quad (8.4.4)$$

Finally,

$$(Y, X) \mapsto \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \quad (8.4.5)$$

where $\alpha = B(X, Y)$ and $\beta = B(Y, Y) + B(X, X)$. The definiteness of B implies that β is nonzero. The results (8.4.3), (8.4.4) and (8.4.5) therefore imply that $(dF_{g,2})_{(X,Y)}$ is surjective. \square

Proof. (Proof of Lemma 8.4.10) The diagonal entries of $F_{g,2}(X, Y)$ are nonzero. Thus, every point in $F_{g,2}^{-1}(F_{g,2}(X, Y))$ must have nonzero components. Moreover, if $B(X, X) \neq B(Y, Y)$, then $B(X', X') \neq B(Y', Y')$ for every $(X', Y') \in F_{g,2}^{-1}(F_{g,2}(X, Y))$. Lemma 8.4.11 then implies that $F_{g,2}(X, Y)$ is a regular value of $F_{g,2}$. \square

8.5 GENERALIZATIONS

8.5.1 LIE ALGEBRAS WITH THE FORM $\mathfrak{g}_1 \oplus \mathfrak{g}_2$

The Automorphism Condition generalizes to Lie algebras with the form $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, where possibly only \mathfrak{g}_1 is simple. But for a slight alteration, the proof of Theorem 8.3.1 applies to this scenario as well. The change that must be made is this: if $\mathfrak{k} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a proper algebra satisfying $\pi_i(\mathfrak{k}) = \mathfrak{g}_i$ for $i = 1, 2$, then $\mathfrak{k} = \text{graph } \theta$ for some surjective homomorphism $\theta : \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$. This is recorded in a corollary.

Corollary 8.5.1. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras with \mathfrak{g}_1 simple, and let $\mathfrak{k} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a subalgebra satisfying $\pi_i(\mathfrak{k}) = \mathfrak{g}_i$ for $i = 1, 2$. Then, \mathfrak{k} is proper if and only if $\mathfrak{k} = \text{graph } \theta$ for some surjective homomorphism $\theta : \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$.*

Proof. As noted above, the essential constituents of this proof are contained in the proof of Lemma 8.3.4. The difference here is that the role of Corollary 8.2.12 is now played by Lemma 8.2.11. Let the ideals $\mathfrak{s}_{\mathfrak{k},1}$ and $\mathfrak{s}_{\mathfrak{k},2}$ be defined as before, namely $\mathfrak{s}_{\mathfrak{k},i} = \ker \pi_{\bar{i}}|_{\mathfrak{k}}$, where \bar{i} denotes the element of $\{1, 2\}$ that is not i . If $\dim \mathfrak{k} > \dim \mathfrak{g}_2$, then, $\ker \pi_2|_{\mathfrak{k}} \neq 0$. It follows, as it did in the proof of Lemma 8.3.4, that $\mathfrak{s}_{\mathfrak{k},1} \neq 0$ and therefore that $\mathfrak{s}_{\mathfrak{k},1} = \mathfrak{g}_1$. If \mathfrak{k} satisfies $\pi_i(\mathfrak{k}) = \mathfrak{g}_i$ for $i = 1, 2$ but is proper, it must therefore be the case that $\dim \mathfrak{k} = \dim \mathfrak{g}_2$. This, according to Lemma 8.2.11, implies that $\mathfrak{k} = \text{graph } \theta$ for some homomorphism $\theta : \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$. θ must be surjective, since $\pi_1(\text{graph } \theta) = \pi_1(\mathfrak{k}) = \mathfrak{g}_1$. \square

Any surjective homomorphism from a simple Lie algebra to another Lie algebra is necessarily a Lie algebra *isomorphism*. Thus, if \mathfrak{g}_2 is also simple, Corollary 8.5.1 yields this,

Corollary 8.5.2. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras with \mathfrak{g}_1 simple, and let $\mathfrak{k} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a subalgebra satisfying $\pi_i(\mathfrak{k}) = \mathfrak{g}_i$ for $i = 1, 2$. Then, \mathfrak{k} is proper if and only if $\mathfrak{k} = \text{graph } \theta$ for some Lie algebra isomorphism $\theta : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$.*

There are at least two situations in which there can fail to be a surjective homomorphism from \mathfrak{g}_2 to \mathfrak{g}_1 : if $\dim \mathfrak{g}_2 < \dim \mathfrak{g}_1$, and if \mathfrak{g}_2 is a semisimple Lie

algebra in which there is no ideal that is isomorphic to \mathfrak{g}_1 (see Lemma 8.5.9 below for the second of these). Corollary 8.5.1 has the following control-theoretic consequences.

Corollary 8.5.3. *Suppose that the system (8.1.1) is defined on $\mathcal{G}_1 \times \mathcal{G}_2$ with \mathcal{G}_1 simple. Suppose additionally that either $\dim \mathcal{G}_1 > \dim \mathcal{G}_2$, or that \mathcal{G}_2 is semisimple but $\text{Lie}(\mathcal{G}_2)$ contains no ideal that is isomorphic to $\text{Lie}(\mathcal{G}_1)$. Then, the component-wise controllability of system (8.1.1) is a necessary and sufficient condition for the controllability of system (8.1.1).*

8.5.2 DIRECT SUMS WITH MORE THAN TWO SUMMANDS

A version of the Automorphism Condition applies to direct sums of Lie algebras with more than two summands. Given a direct sum $\bigoplus_{i=1}^n \mathfrak{g}_i$ and a subset $I \subset \{1, \dots, n\}$, let $\pi_I : \bigoplus_{i=1}^n \mathfrak{g}_i \rightarrow \bigoplus_{i \in I} \mathfrak{g}_i$ denote the obvious projection map. The generalization of Theorem 8.3.1 is this,

Theorem 8.5.4. *Let \mathfrak{g}_i be a simple Lie algebra for $i = 1, \dots, m$. Let $\mathfrak{k} \subset \bigoplus_{i=1}^n \mathfrak{g}_i$ be a subalgebra satisfying $\pi_i(\mathfrak{k}) = \mathfrak{g}_i$ for $i = 1, \dots, m$. Then, \mathfrak{k} is a proper subalgebra if and only if for some distinct indices $j, k \in \{1, \dots, n\}$, $\pi_{\{j,k\}}(\mathfrak{k})$ is a proper subalgebra of $\mathfrak{g}_j \oplus \mathfrak{g}_k$.*

Remark 8.5.5. It follows from Corollary 8.5.1 that, in the scenario described in Theorem 8.5.4, if \mathfrak{k} is a proper subalgebra, then, $\pi_{\{j,k\}}(\mathfrak{k}) = \text{graph } \theta$ for some isomorphism $\theta : \mathfrak{g}_j \rightarrow \mathfrak{g}_k$, where j and k are the indices provided by the theorem.

■

Remark 8.5.6. In control-theoretic terms, Theorem 8.5.4 says that the component-wise controllability of a driftless, left-invariant bilinear system on $\mathcal{G}_1 \times \dots \times \mathcal{G}_n$, where each \mathcal{G}_i is simple, is a necessary and sufficient condition for the controllability of the full system, provided that no two component systems are related by an isomorphism. ■

The proof of Theorem 8.5.4 requires a couple of lemmas and some notation. For each subset $I \subset \{1, \dots, n\}$, let \mathfrak{g}_I denote $\bigcap_{i \in I^c} \ker \pi_i$, the set of elements

whose i^{th} component is zero for each $i \in I^c$. For example, $\mathfrak{g}_{\{i\}}$ is the image of the inclusion $\mathfrak{g}_i \hookrightarrow \bigoplus_{i=1}^n \mathfrak{g}_i$ (note that $\mathfrak{g}_{\{i\}}$ is an ideal in $\bigoplus_{j=1}^n \mathfrak{g}_j$). Also, observe that $\ker \pi_{\{i\}^c} = \mathfrak{g}_{\{i\}}$ and that $\mathfrak{g}_I \cong \bigoplus_{i=1}^n \mathfrak{g}_i / \mathfrak{g}_{I^c}$.

Lemma 8.5.7. *Let \mathfrak{g}_i be simple Lie algebras for $i = 1, \dots, n$. If $J \subset \bigoplus_{i=1}^n \mathfrak{g}_i$ is an ideal, then $J = \mathfrak{g}_I$ for some $I \subset \{1, \dots, n\}$.*

Proof. The simplicity of the \mathfrak{g}_i implies that $J \cap \mathfrak{g}_{\{i\}}$ is equal either to 0 or to $\mathfrak{g}_{\{i\}}$. Let $I \subset \{1, \dots, n\}$ denote the set of indices i for which $J \cap \mathfrak{g}_{\{i\}} = \mathfrak{g}_{\{i\}}$. Then, $J = \mathfrak{g}_I$. \square

The next lemma generalizes Lemma 8.3.9 and Corollary 8.3.10.

Lemma 8.5.8. *Let \mathfrak{g}_i and \mathfrak{k} be as they are in Theorem 8.5.4 and let $i \in \{1, \dots, n\}$. Then, $\pi_i(\mathfrak{g}_{\{i\}} \cap \mathfrak{k})$ is equal to either 0 or \mathfrak{g}_i .*

Proof. $\mathfrak{g}_{\{i\}}$ is an ideal in $\bigoplus_{j=1}^n \mathfrak{g}_j$. Thus, $\mathfrak{g}_{\{i\}} \cap \mathfrak{k}$ is an ideal in \mathfrak{k} . Lemma 8.3.8 and the surjectivity of $\pi_i|_{\mathfrak{k}}$ imply that $\pi_i(\mathfrak{g}_{\{i\}} \cap \mathfrak{k})$ is an ideal in \mathfrak{g}_i . This proves the lemma. \square

Lemma 8.5.9. *Let \mathfrak{g} be a simple Lie algebra. If $\theta : \bigoplus_{i=1}^n \mathfrak{g}_i \rightarrow \mathfrak{g}$ is a surjective homomorphism, then some for $i \in \{1, \dots, n\}$, $\ker \theta = \ker \pi_i$ and $\theta|_{\mathfrak{g}_{\{i\}}} : \mathfrak{g}_{\{i\}} \rightarrow \mathfrak{g}$ is an isomorphism.*

Proof. By Lemma 8.5.7, $\ker \theta = \mathfrak{g}_I$ for some subset $I \subset \{1, \dots, n\}$. The quotient map $\bar{\theta} : \bigoplus_{i=1}^n \mathfrak{g}_i / \mathfrak{g}_I \cong \mathfrak{g}_{I^c} \rightarrow \mathfrak{g}$ is then an isomorphism. The simplicity of \mathfrak{g} implies that $\mathfrak{g}_{I^c} = \bigoplus_{j \in I^c} \mathfrak{g}_j$ cannot have more than one summand. It follows that, $I^c = \{i\}$ for some $i \in \{1, \dots, n\}$. Thus, $\ker \theta = \mathfrak{g}_{\{i\}^c} = \ker \pi_i$. \square

Proof. (Proof of Theorem 8.5.4) If, for some distinct $i, j \in \{1, \dots, n\}$, $\pi_{\{i,j\}}(\mathfrak{k}) \subset \mathfrak{g}_i \oplus \mathfrak{g}_j$ is proper, then \mathfrak{k} must be proper. For the converse, assume that \mathfrak{k} is a proper subalgebra satisfying $\pi_i(\mathfrak{k}) = \mathfrak{g}_i$ for $i = 1, \dots, n$. The proof proceeds by induction on n , the number of Lie algebras \mathfrak{g}_i . The case $n = 2$ is dispatched by Corollary 8.5.1. Suppose now that $n > 2$ and that the theorem is true for all $p \in \mathbb{Z}$ with $0 \leq p \leq n - 1$.

Let \mathfrak{k}' be the image $\pi_{\{n\}^c}(\mathfrak{k})$ in $\oplus_{i=1}^{n-1} \mathfrak{g}_i$. Like \mathfrak{k} , \mathfrak{k}' satisfies $\pi_i(\mathfrak{k}') = \mathfrak{g}_i$ for $i = 1, \dots, n-1$.

\mathfrak{k}' is either proper, or it is not. If \mathfrak{k}' is proper, then, by the inductive assumption, $\pi_{\{j,k\}}(\mathfrak{k}')$ is a proper subalgebra of $\mathfrak{g}_j \oplus \mathfrak{g}_k$ for some distinct $j, k \in \{1, \dots, n-1\}$. This completes the inductive step in the event that \mathfrak{k}' is proper, since $\pi_{\{j,k\}}(\mathfrak{k}') = \pi_{\{j,k\}}(\mathfrak{k})$.

Suppose now that \mathfrak{k}' is not proper (i.e., $\mathfrak{k}' = \oplus_{j=1}^{n-1} \mathfrak{g}_j$, or equivalently, $\pi_{\{n\}^c}|_{\mathfrak{k}}$ is surjective), and consider the kernel $\ker \pi_{\{n\}^c}|_{\mathfrak{k}}$.

Claim. If \mathfrak{k}' is not proper, then $\ker \pi_{\{n\}^c}|_{\mathfrak{k}} = \{0\}$.

To see why this is, note that $\ker \pi_{\{n\}^c}|_{\mathfrak{k}} = \mathfrak{k} \cap \ker \pi_{\{n\}^c} = \mathfrak{k} \cap \mathfrak{g}_{\{n\}}$. So, if $\ker \pi_{\{n\}^c}|_{\mathfrak{k}} \neq \{0\}$, then $\mathfrak{k} \cap \mathfrak{g}_{\{n\}} \neq \{0\}$. But in this case Lemma 8.5.8 implies that $\pi_n(\mathfrak{k} \cap \mathfrak{g}_{\{n\}}) = \mathfrak{g}_n$ and therefore that $\mathfrak{g}_{\{n\}} \subset \mathfrak{k}$. But, the conditions $\mathfrak{g}_{\{n\}} \subset \mathfrak{k}$ and $\pi_{\{n\}^c}(\mathfrak{k}) = \oplus_{j=1}^{n-1} \mathfrak{g}_j$ imply that $\mathfrak{k} = \oplus_{j=1}^n \mathfrak{g}_j$, which is not the case (\mathfrak{k} was assumed to be proper). Thus, by contradiction, the claim must be true.

The claim implies that $\pi_{\{n\}^c}|_{\mathfrak{k}}$ is an isomorphism between \mathfrak{k} and $\mathfrak{k}' = \oplus_{j=1}^{n-1} \mathfrak{g}_j$. By Lemma 8.2.11, $\mathfrak{k} = \text{graph } \theta$ for some homomorphism $\theta : \oplus_{j=1}^{n-1} \mathfrak{g}_j \rightarrow \mathfrak{g}_n$. By Lemma 8.5.9 $\theta|_{\mathfrak{g}_i} : \mathfrak{g}_i \rightarrow \mathfrak{g}_n$ is an isomorphism for some $i \in \{1, \dots, n-1\}$. Thus, $\pi_{\{i,n\}}(\mathfrak{k}) = \text{graph } \theta|_{\mathfrak{g}_i} \neq \mathfrak{g}_i \oplus \mathfrak{g}_n$. This proves the inductive step in the case that \mathfrak{k}' is not proper and completes the proof of Theorem 8.5.4. □

8.5.3 GENERALIZING COROLLARY 8.4.7

Section 8.4 began with this admission: by itself, the Automorphism Condition is not a means by which the controllability of a given coupled system can be determined. The Automorphism Condition only identifies what such a means must do: it must distinguish between distinct orbits of the diagonal action of $\text{Aut}(\mathfrak{g})$ on $\oplus_{i=1}^m \mathfrak{g}$. This task is accomplished in Section 8.4 with the tacit help of *automorphism-invariant functions*.

Definition 8.5.10. Let \mathfrak{g} be a Lie algebra and let V be a real, finite-dimensional vector

space. A function $F : \oplus_{i=1}^m \mathfrak{g} \rightarrow V$ is **automorphism-invariant** if $F(\theta X_1, \dots, \theta X_m) = F(X_1, \dots, X_m)$ for all $\theta \in \text{Aut}(\mathfrak{g})$ and $(X_1, \dots, X_m) \in \oplus_{i=1}^m \mathfrak{g}$.

Example 8.5.11. The Killing form $B : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathbb{R}$ on \mathfrak{g} is an automorphism-invariant function, as is the function $F_{\mathfrak{g},m} : \oplus_{i=1}^m \mathfrak{g} \rightarrow \text{Sym}_m$ defined in Remark 8.4.9.

A function is automorphism-invariant if and only if it is constant on the orbits of \mathcal{A}_D . Thus, if $(X_1, \dots, X_m), (Y_1, \dots, Y_m) \in \oplus_{i=1}^m \mathfrak{g}$ and $F(X_1, \dots, X_m) \neq F(Y_1, \dots, Y_m)$ for some automorphism-invariant function F , then (X_1, \dots, X_m) and (Y_1, \dots, Y_m) are *not* in the same orbit of \mathcal{A}_D .

Automorphism-invariant functions therefore provide a means of distinguishing between the orbits of \mathcal{A}_D . In doing so, automorphism-invariant functions provide a means for determining the controllability of system (8.1.1): system (8.1.1) is controllable if it is component-wise controllable and if there exists an automorphism invariant function satisfying $F(X_1, \dots, X_m) \neq F(Y_1, \dots, Y_m)$.

If $F : \oplus_{i=1}^m \mathfrak{g} \rightarrow V$ is automorphism-invariant, then, for all $p \in V$, $F^{-1}(p)$ is either empty or a union of orbits of \mathcal{A}_D . If F has the property that, for all $p \in V$, $F^{-1}(p)$ is either empty or a *single* orbit of \mathcal{A}_D then (X_1, \dots, X_m) and $(Y_1, \dots, Y_m) \in \oplus_{i=1}^m \mathfrak{g}$ are in the same orbit of the diagonal action if and only if $F(X_1, \dots, X_m) = F(Y_1, \dots, Y_m)$. The problem of distinguishing between orbits of \mathcal{A}_D is, in this case, reduced to calculating F .

Definition 8.5.12. Let $S \subset \oplus_{i=1}^m \mathfrak{g}$ be a subset. An automorphism-invariant function $F : \oplus_{i=1}^m \mathfrak{g} \rightarrow V$ is **orbit-separating** on S if for all $(X_1, \dots, X_m), (Y_1, \dots, Y_m) \in S$, $F(X_1, \dots, X_m) = F(Y_1, \dots, Y_m)$ if and only if (X_1, \dots, X_m) and (Y_1, \dots, Y_m) are in the same orbit of \mathcal{A}_D .

Remark 8.5.13. It may suffice in some situations to determine whether two points from a given subset $S \subset \oplus_{i=1}^m \mathfrak{g}$ are in the same orbit of \mathcal{A}_D .

For example, automorphism-invariant functions were introduced for the purpose of determining the controllability of system (8.1.1). However, if system (8.1.1) is not component-wise controllable, there is no need to find an automorphism-invariant function satisfying $F(X_1, \dots, X_m) \neq F(Y_1, \dots, Y_m)$

(where $(X_1, \dots, X_m), (Y_1, \dots, Y_m) \in \oplus_{i=1}^m \mathfrak{g}$ are the points defining system (8.1.1)). Thus, automorphism-invariant functions being used for this purpose need not be orbit-separating on all of $\oplus_{i=1}^m \mathfrak{g}$. They need only be so on the set $S = \{(X_1, \dots, X_m) \in \oplus_{i=1}^m \mathfrak{g} : \langle \{X_1, \dots, X_m\} \rangle = \mathfrak{g}\}$, the set of points whose components generate \mathfrak{g} . ■

Example 8.5.14. Corollary 8.4.7 provides an example of an orbit-separating function, $F_{\mathfrak{so}(3), 2}$, on $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Remark 8.4.9 introduced a generalization of $F_{\mathfrak{so}(3), 2}$, $F_{\mathfrak{g}, m}$, defined on the m -fold sum $\mathfrak{g} \oplus \dots \oplus \mathfrak{g}$. Unfortunately, $F_{\mathfrak{g}, m}$ is not necessarily orbit-separating. To see this, suppose that $m = 2$, that $\dim \mathfrak{g} > 3$. Suppose furthermore that \mathfrak{g} is compact and semisimple (so that the Killing form on \mathfrak{g} is negative definite). Note that the semisimplicity of \mathfrak{g} implies that $\dim \text{Aut}(\mathfrak{g}) = \dim \mathfrak{g}$, where $\dim \text{Aut}(\mathfrak{g})$ refers to the dimension of $\text{Aut}(\mathfrak{g})$ as a Lie group. It follows that the dimension of the orbit of \mathcal{A}_D through any point in $\mathfrak{g} \oplus \mathfrak{g}$ cannot exceed $\dim \text{Aut}(\mathfrak{g}) = \dim \mathfrak{g}$.

Under these assumptions, $F_{\mathfrak{g}, 2} : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \text{Sym}_2$ is subject to Lemma 8.4.10. Let $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ be such that $X \neq 0$ and $Y \neq 0$ and let $p = F_{\mathfrak{g}, 2}(X, Y)$. According to Lemma 8.4.10, $F_{\mathfrak{g}, 2}^{-1}(p)$ is a submanifold of $\mathfrak{g} \oplus \mathfrak{g}$ whose dimension is $2 \dim \mathfrak{g} - 3$. Given the assumption that $\dim \mathfrak{g} > 3$, $2 \dim \mathfrak{g} - 3$ is strictly greater than $\dim \mathfrak{g}$. It now follows that $F_{\mathfrak{g}, 2}^{-1}(p)$ cannot be a single orbit of \mathcal{A}_D , since the dimension of any such orbit was just shown to be less than or equal to $\dim \mathfrak{g}$.

No attempt is made here to study when, if ever, $F_{\mathfrak{g}, m}$ is orbit separating. Instead, the remainder of this section discusses the possibility of augmenting $F_{\mathfrak{g}, 2}$ in such a way that it becomes orbit-separating. The discussion begins by establishing conditions under which the preimage of a given point under an automorphism-invariant function is a single orbit. The results of this discussion will guide the search for an automorphism-invariant function that extends $F_{\mathfrak{g}, 2}$. This is all done under two simplifying assumptions : that $\dim \text{Aut}(\mathfrak{g}) = \dim \mathfrak{g}$ (this is the case if, for example, \mathfrak{g} is semisimple), and that $\text{Aut}(\mathfrak{g})$ is compact and connected (this is true for $\mathfrak{g} = \mathfrak{so}(n)$ with $n \geq 5$). Under the second assumption the orbits of the diagonal action of $\text{Aut}(\mathfrak{g})$ on $\mathfrak{g} \oplus \mathfrak{g}$ are all compact, connected

embedded submanifolds.

Two preliminary observations, 8.5.15 and 8.5.17 below, are needed before the discussion can commence. The first is provided without proof. The second is dicussed in [43].

Proposition 8.5.15. *Let $\theta \in \text{Aut}(\mathfrak{g})$ be an automorphism, $S \subset \mathfrak{g}$ a subset and suppose that $\theta X = X$ for all $X \in S$. $\theta|_{\langle S \rangle} : \langle S \rangle \rightarrow \mathfrak{g}$, the restriction of θ to the subalgebra generated by S , is then just the inclusion of $\langle S \rangle$ into \mathfrak{g} .*

Suppose now that $X, Y \in \mathfrak{g}$ are such that $\langle \{X, Y\} \rangle = \mathfrak{g}$. It follows from Proposition 8.5.15 that the only automorphism that fixes both X and Y is the identity of \mathfrak{g} . In terms of the diagonal action, if $\langle \{X, Y\} \rangle = \mathfrak{g}$, then the stabilizer of $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ is the identity in $\text{Aut}(\mathfrak{g})$. Thus, if X and Y generate \mathfrak{g} , the dimension of the orbit $\mathcal{O}_{(X,Y)}$ of \mathcal{A}_D is equal to that of $\text{Aut}(\mathfrak{g})$, which in turn is equal to $\dim \mathfrak{g}$, given the assumptions made above.

Remark 8.5.16. Henceforth, the set of pairs $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ satisfying $\langle \{X, Y\} \rangle = \mathfrak{g}$ will be denoted by S_{gen} . When \mathfrak{g} is semisimple, S_{gen} is open and dense in $\mathfrak{g} \oplus \mathfrak{g}$ [26].

Proposition 8.5.17. *Let M be a real, finite-dimensional smooth manifold and let $N, P \subset M$ be two submanifolds of M (not necessarily embedded) such that $N \subset P$. If P is an embedded submanifold of M , then N is a submanifold of P .*

Returning now to the problem of finding conditions under which the preimage of a point under an automorphism-invariant function is a single orbit, let $F : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathbb{R}^{\dim \mathfrak{g}}$ be an automorphism-invariant function and suppose that $p \in \mathbb{R}^{\dim \mathfrak{g}}$ is a regular value of F whose preimage is not empty. $F^{-1}(p)$ is then an embedded submanifold of $\mathfrak{g} \oplus \mathfrak{g}$ whose dimension is $\dim \mathfrak{g}$.

Suppose now that $F^{-1}(p)$ contains a point $(X, Y) \in S_{\text{gen}}$. The automorphism-invariance of F implies that $\mathcal{O}_{(X,Y)}$ is contained in $F^{-1}(p)$. By Proposition 8.5.17, $\mathcal{O}_{(X,Y)}$ is a submanifold of $F^{-1}(p)$ whose dimension, by the remarks following Proposition 8.5.15, is $\dim \mathfrak{g}$.

The equality $\dim F^{-1}(p) = \dim \mathcal{O}_{(X,Y)} = \dim \mathfrak{g}$ implies that $\mathcal{O}_{(X,Y)}$ is an open subset of $F^{-1}(p)$. The compactness of $\mathcal{O}_{(X,Y)}$ implies that it is a closed subset of $F^{-1}(p)$. If $F^{-1}(p)$ is connected, it would finally follow that $F^{-1}(p) = \mathcal{O}_{(X,Y)}$.

Weaving the foregoing thread of assumptions together yields the following observation.

Observation 8.5.18. *Let \mathfrak{g} be a Lie algebra whose automorphism group is compact and connected and satisfies $\dim \text{Aut}(\mathfrak{g}) = \dim \mathfrak{g}$. Let S_{gen} be the subset of $\mathfrak{g} \oplus \mathfrak{g}$ consisting of those pairs (X, Y) satisfying $\langle \{X, Y\} \rangle = \mathfrak{g} \oplus \mathfrak{g}$. Finally, let $F : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathbb{R}^{\dim \mathfrak{g}}$ be a function with the following property : for every $(X, Y) \in S_{\text{gen}}$, $F(X, Y)$ is a regular value of F whose preimage is connected. Then, F is orbit-separating on S_{gen} .*

Let $F_{\mathfrak{g},2}$ be thought of as mapping into \mathbb{R}^3 , instead of Sym_2 (say, by $F(X, Y) = (\|X\|^2, \langle X, Y \rangle, \|Y\|^2)$). Observation 8.5.18 raises the hope of converting $F_{\mathfrak{g},2}$ into an orbit-separating function on S_{gen} by appending to it a judiciously chosen automorphism-invariant function $H : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathbb{R}^{\dim \mathfrak{g}-3}$. The hope, more precisely, is that the conjoined function $(X, Y) \mapsto (F_{\mathfrak{g},2}(X, Y), H(X, Y)) \in \mathbb{R}^{\dim \mathfrak{g}}$ will be orbit-separating, at least on S_{gen} .

The Baker-Campbell-Hausdorff series is a spring of automorphism-invariant functions on $\mathfrak{g} \oplus \mathfrak{g}$. To be precise, let $H_n(X, Y)$ denote the n^{th} homogeneous term in the Baker-Campbell-Hausdorff series for $X, Y \in \mathfrak{g}$. Then, with B still denoting the Killing form on \mathfrak{g} , $(X, Y) \mapsto B(H_n(X, Y), H_n(X, Y))$ is an automorphism-invariant function.

Remark 8.5.19. For notational convenience the Killing form on \mathfrak{g} is denoted by \langle, \rangle in the following conjecture, rather than the usual B . For $X \in \mathfrak{g}$, $\|X\|^2$ will denote $\langle X, X \rangle$. ■

Conjecture 8.5.20. *Let \mathfrak{g} be a Lie algebra satisfying the conditions of Observation*

8.5.18. Let $H : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathbb{R}^{\dim \mathfrak{g}}$ be the function

$$(X, Y) \mapsto \begin{pmatrix} \|X\|^2 \\ \langle X, Y \rangle \\ \|Y\|^2 \\ \|H_2(X, Y)\|^2 \\ \vdots \\ \|H_{\dim \mathfrak{g}-2}(X, Y)\|^2 \end{pmatrix}$$

is orbit-separating on S_{gen} .

According to Observation 8.5.18, Conjecture 8.5.20 is true so long as H has the property that for every $(X, Y) \in S_{\text{gen}}$, $H(X, Y)$ is a regular value of H whose preimage is connected.

Remark 8.5.21. In general, H cannot be orbit-separating on all of $\mathfrak{g} \oplus \mathfrak{g}$. If $X, X' \in \mathfrak{g}$ are distinct points such that $\|X\| = \|X'\| := r$, then $H(X, 0) = H(X', 0)$ (all of the non-linear terms of the Baker - Campbell - Hausdorff series for X and Y vanish if X or Y is zero). However, $\text{Aut}(\mathfrak{g})$ does not necessarily act transitively on the set $S_r = \{Z \in \mathfrak{g} : \|Z\| = r\}$. For example, if \mathfrak{g} is compact and semisimple, and \langle, \rangle then an inner product, S_r is a sphere (for $r \neq 0$). Since there are only finitely many Lie groups that act transitively on spheres [34], it follows that H is not orbit-separating on all of $\mathfrak{g} \oplus \mathfrak{g}$.

8.6 APPENDIX: THE AUTOMORPHISM GROUPS OF SEMISIMPLE LIE ALGEBRAS

This appendix describes the structure of $\text{Aut}(\mathfrak{g})$ when \mathfrak{g} is the Lie algebra of a compact, semisimple, connected Lie group \mathcal{G} . In short, $\text{Aut}(\mathfrak{g})$ consists of finitely many cosets of its identity component, and its identity component is $\text{Ad}(\mathcal{G})$, the image of \mathcal{G} under the adjoint map $\text{Ad} : \mathcal{G} \rightarrow \text{Aut}(\mathfrak{g})$. If \mathcal{G} is matrix Lie group, it follows that there are finitely many automorphisms $C_i \in \text{Aut}(\mathfrak{g})$ such that every automorphism of \mathfrak{g} is of the form $X \in \mathfrak{g} \mapsto C_i(\theta X \theta^{-1})$ for some $\theta \in \mathcal{G}$.

In general, $\text{Aut}(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g})$, the group of invertible linear maps from \mathfrak{g} to itself. As such, $\text{Aut}(\mathfrak{g})$ is a Lie group. As a subgroup of $GL(\mathfrak{g})$, the Lie algebra of $\text{Aut}(\mathfrak{g})$ is $\text{Der}(\mathfrak{g})$, the set of derivations of \mathfrak{g} . Thus, the identity component of $\text{Aut}(\mathfrak{g})$ is the unique connected subgroup of $\text{Aut}(\mathfrak{g})$ whose Lie algebra is $\text{Der}(\mathfrak{g})$.

$\text{Der}(\mathfrak{g})$ contains the subalgebra $\text{ad}(\mathfrak{g})$, the image of \mathfrak{g} under its adjoint map $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$. $\text{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ is the differential of the adjoint map $\text{Ad} : \mathcal{G} \rightarrow \text{Aut}(\mathfrak{g})$. The connectedness of \mathcal{G} implies that $\text{Ad}(\mathcal{G})$ is the connected subgroup of $\text{Aut}(\mathfrak{g})$ whose Lie algebra is $\text{ad}(\mathfrak{g})$.

When \mathfrak{g} is semisimple, $\text{Der}(\mathfrak{g})$ is in fact equal to $\text{ad}(\mathfrak{g})$ (see Proposition 1.12.1 in [25]). It follows that, when \mathfrak{g} is Lie algebra of a semisimple group \mathcal{G} , the identity component of $\text{Aut}(\mathfrak{g})$ is $\text{Ad}(\mathcal{G})$.

The identity component of any Lie group is a normal subgroup. Thus, $\text{Aut}(\mathfrak{g})/\text{Ad}(\mathcal{G})$ is a group. The structure of the quotient group $\text{Aut}(\mathfrak{g})/\text{Ad}(\mathcal{G})$ for compact semisimple \mathcal{G} is described in Theorem 7.7 of [25].

Theorem 8.6.1. (*Theorem 7.8 of [25]*) *If \mathfrak{g} is the Lie algebra of a compact semisimple Lie group \mathcal{G} then $\text{Aut}(\mathfrak{g})/\text{Ad}(\mathcal{G})$ is isomorphic to the group of automorphisms of the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$, the complexification of \mathfrak{g} .*

Remark 8.6.2. The Dynkin diagrams of all the complex semisimple Lie algebras can be found in [25].

Corollary 8.6.3. $\text{Aut}(\mathfrak{so}(3)) = \text{Ad}(SO(3))$. *Every automorphism of $\mathfrak{so}(3)$ is of the form $X \in \mathfrak{so}(3) \mapsto UXU^T$, with $U \in SO(3)$.*

Corollary 8.6.4. $\text{Aut}(\mathfrak{su}(2)) = \text{Ad}(SU(2))$. *Every automorphism of $\mathfrak{su}(2)$ is therefore of the form $X \in \mathfrak{su}(2) \mapsto UXU^\dagger$, with $U \in SU(2)$.*

Proof. The complexification of both $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ is $\mathfrak{su}(2, \mathbb{C})$. The Dynkin diagram of $\mathfrak{su}(2, \mathbb{C})$ is the graph with one vertex; its automorphism group therefore consists only of the identity. The corollaries now follow from Theorem 8.6.1. □

The remainder of this appendix describes the automorphism groups of $\mathfrak{so}(n)$ (for $n \geq 5$) and $\mathfrak{su}(n)$.

The Dynkin diagrams of the complexifications of the odd-dimensional special orthogonal algebras $\mathfrak{so}(2n+1)$, $n \geq 1$, have trivial automorphism groups.

Theorem 8.6.1 implies that $\text{Aut}(\mathfrak{so}(2n+1)) = \text{Ad}(SO(2n+1))$.

The Dynkin diagrams of the complexifications of the even-dimensional special orthogonal algebras $\mathfrak{so}(2n)$ have nontrivial automorphism groups: they are all equal to \mathbb{Z}_2 . $\text{Aut}(\mathfrak{so}(2n))$ therefore has two connected components, one of which is the identity component $\text{Ad}(SO(2n))$.

To find the non-identity component of $\text{Aut}(\mathfrak{so}(2n))$ note that $\text{Ad} : SO(2n) \rightarrow \text{Aut}(\mathfrak{so}(2n))$ is actually the restriction to $SO(2n)$ of the adjoint map of $O(2n)$. The kernel of $\text{Ad} : O(2n) \rightarrow \text{Aut}(\mathfrak{so}(2n))$ is $\{I, -I\}$, where I denotes the $2n \times 2n$ identity matrix. By virtue of $2n$ being even, $\{I, -I\}$ is contained in $SO(2n)$.

Let U_\circ be an element of $O(2n) - SO(2n)$.

Claim 8.6.5. *There is no $U \in SO(2n)$ for which $\text{Ad}(U_\circ) = \text{Ad}(U)$.*

If there were such a $U \in SO(2n)$, $U_\circ U^{-1}$ would be in $\ker \text{Ad}$, and therefore in $SO(2n)$. However, $U_\circ U^{-1}$ is in the coset $U_\circ \cdot SO(2n)$, which is equal to $O(2n) - SO(2n)$. This is a contradiction. No such U exists.

It now follows that $\text{Ad}(U_\circ) \cdot \text{Ad}(SO(2n))$ is the non-identity component of $\text{Aut}(\mathfrak{so}(2n))$. Every element of $\text{Aut}(\mathfrak{so}(2n))$ is equal to either $\text{Ad}(U)$ for some $U \in SO(2n)$, or to $\text{Ad}(U_\circ U) = \text{Ad}(U_\circ) \cdot \text{Ad}(U)$ for some $U \in SO(2n)$. Automorphisms with either form preserve the eigenvalues of their arguments.

Corollary 8.6.6. *Let $\theta \in \text{Aut}(\mathfrak{so}(n))$. For every $X \in \mathfrak{so}(n)$, the eigenvalues of θX are equal to those of X .*

The complexification of $\mathfrak{su}(n)$ is $\mathfrak{sl}(n, \mathbb{C})$. The Dynkin diagram of $\mathfrak{sl}(n, \mathbb{C})$ has one nontrivial automorphism. It follows that $\text{Aut}(\mathfrak{su}(n))$ has two components. The identity component of $\text{Aut}(\mathfrak{su}(n))$ is $\text{Ad}(SU(n))$. To describe the remaining component of $\text{Aut}(\mathfrak{su}(n))$ it suffices, as it did for $\mathfrak{so}(n)$, to find an automorphism,

θ_o , that is not in the identity component $\text{Ad}(SU(n))$. The non-identity component of $\text{Aut}(\mathfrak{su}(n))$ is then the coset $\theta_o \cdot \text{Ad}(SU(n))$.

Entry-wise complex conjugation defines a map $C : \mathfrak{su}(n) \rightarrow \mathfrak{su}(n)$. C is in $\text{Aut}(\mathfrak{su}(n))$. However, C is not in $\text{Ad}(SU(n))$. For all $X \in \mathfrak{su}(n)$, the eigenvalues of CX are the complex conjugates of those of X , whereas the eigenvalues of $\text{Ad}(U)X$ are the same as those of X for every $U \in SU(n)$.

It now follows that every automorphism in $\text{Aut}(\mathfrak{su}(n))$ is equal either to $\text{Ad}(U)$ for some $U \in SU(n)$, or to $C \cdot \text{Ad}(U)$, for some $U \in SU(n)$.

Corollary 8.6.7. *Let $\theta \in \text{Aut}(\mathfrak{su}(n))$. Either the eigenvalues of θX are the same as those of X for all $X \in \mathfrak{su}(n)$, or the eigenvalues of θX are the complex conjugates of those of X for all $X \in \mathfrak{su}(n)$.*

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